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Exercise 11
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SS 09

Exercises on Theoretical Astroparticle Physics

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1. **Scalar Field Inflation**
From the gravitational field equations for the Friedmann Universe (with $G = 1$)
$$\ddot{a} = -\frac{4\pi}{3}(\rho + 3p)a$$
we see that for an accelerated expansion of the Universe we need $\rho + 3p < 0$. This can e.g. be realized by a vacuum constant with $p = -\rho$. Let's see if we can realize this with one scalar field φ.

(a) Assume a scalar field in a gravitational dynamic background with action
$$S_\varphi = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right).$$
Compute the energy momentum tensor $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g_{\mu\nu}}$ and identify the energy density ρ and pressure p under the assumption of spatial homogenity. Under which condition will the Universe inflate?

(b) Start from the first law of thermodynamics $\mathrm{d}E = -p\mathrm{d}V$ to obtain the e.o.m. for φ:
$$\ddot{\varphi} - 3H\dot{\varphi} + \frac{\partial V}{\partial\varphi} = 0$$

(c) Now we try the simplest choice for a nontrivial scalar potential: $V(\varphi) = \frac{1}{2}m^2\varphi^2$. Take the flat space Friedmann equation $H^2 = \frac{8\pi}{3}\rho$ to eliminate the Hubble parameter. Eliminate also the time dependence by introducing $\dot{\varphi}(\varphi)$ and obtain the differential equation:
$$\frac{\mathrm{d}\dot{\varphi}}{\mathrm{d}\varphi} = \sqrt{12\pi}\left(\dot{\varphi}^2 + m^2\varphi^2\right)^{1/2} + m^2\frac{\varphi}{\dot{\varphi}}$$

(d) Take the limit $|\dot{\varphi}| \gg m\varphi$ to simplify the equation. Solve it for $\dot{\varphi}$ to get a differential equation for $\varphi(t)$. Its solution can be inserted into the Friedmann equation to find the relation $a \propto t^{1/3}$. Hence we see that in this case we don't find inflation. (Why was this clear from the beginning?)

(e) From Fig.1 we see that there exists an attractor solution to which the previously found solution quickly converges. Due to the flatness of the attractor trajectory we can assume $\frac{\mathrm{d}\dot{\varphi}}{\mathrm{d}\varphi} = 0$. To obtain a inflating solution we further assume $m\varphi \gg |\dot{\varphi}|$. Show that this leads to $\dot{\varphi} = -\frac{m}{\sqrt{12\pi}}$. Now via the Friedmann equation your should find that this allows for an inflationaty solution
$$a(t) = a_f e^{-\frac{m^2}{6}(t_f - t)^2}, .$$

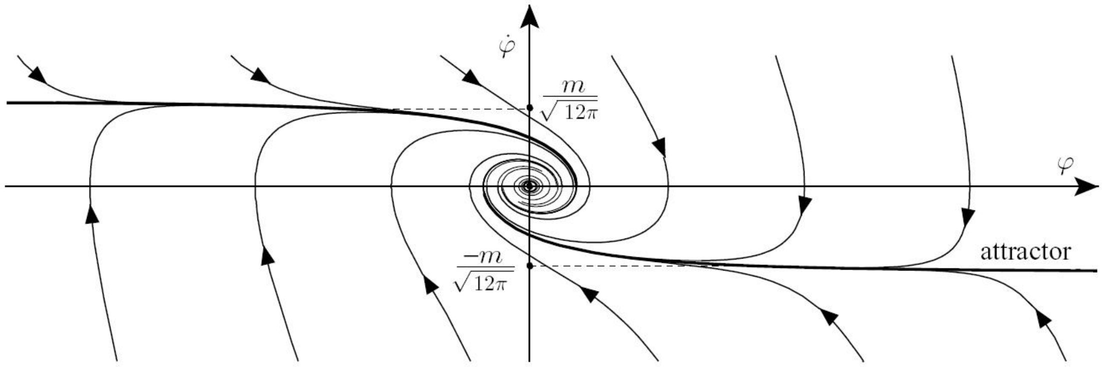

Figure 1: $\dot{\varphi}(\varphi)$ flow

(f) The whole approximation is only valid for curvatures and hence energy densities below the Planck scale. Otherwise the effects of quantum gravity will become imporant. Assuming the inflaton mass to be $m \propto 10^{13}$GeV and the inflation to start in the validity regime (sub Planckian) of this example, by which factor does the universe inflate during which time?

2. **Metric Pertubations**
The most promising way to describe structures (inhomogenities) in our Universe is by early quantum fluctuations which were blown up to cosmological scales by inflation. These fluctuations can be described by metric pertubations around some fixed background value.

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}$$

For convenience we use conformal time η such that the background metric takes the form

$$g^0_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \, .$$

Since we assume spatial homogenity and isotropy we can disjoint the pertubations into irreducible parts under spatial rotations,

$$\begin{aligned} \delta g_{00} &= 2a^2\phi \\ \delta g_{oi} &= a^2\left(B_{,i} + S_i\right) \\ \delta g_{ij} &= a^2\left(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}\right) \end{aligned}$$

where ϕ, B, ψ and E are scalars, S_i and F_i are divergenceless vectors (${S^i}_{,i} = {F^i}_{,i} = 0$) and h_{ij} is a symmetric tensor satisfying $h^i_i = 0$, $h^i_{j,i} = 0$.

(a) Count the various degrees of freedom of the irreducible pertubations. Does it agree with your expectations?

(b) We know that not all of these dof's are physical since we have the freedom to choose a coordinate system. Let's perform a small coordinate transformation $\delta x^\mu = \xi^\mu(x^\mu)$. How does the pertubation $\delta g_{\mu\nu}$ transform?

(c) We can also disjoint the transformation parameter ξ^μ into irreducible parts like this: $\xi^\mu = (\xi^0, \xi^i)$ with $\xi^i = \xi^i_\perp + \zeta^{,i}$ satisfying $\xi^i_{\perp,i} = 0$. How do the irreducible parts of $\delta g_{\mu\nu}$ transform now?

(d) Use the transformation behaviours to construct the invariant quantities $\Psi = \psi + f(E, B, a)$, $\Phi = \phi + g(E, B, a)$ and $V_i(S_i, F_i)$. h_{ij} is already invariant and describes the well known gravitational waves.