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Exercise 4
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SS 09

Exercises on Theoretical Astroparticle Physics

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We look at the instantons and axions...

1. **Instantons**

   We consider instantons of $SU(2)$ gauge theories in the four-dimensional euclidean space $M = \mathbb{R}^4$. The action is given by

   $$S^E_{YM} = -\frac{1}{2g^2}\int_M d^4x \,\mathrm{tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \tag{1}$$

   where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu A_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$. The gauge field $\mathcal{A}_\mu$ and the field strength $\mathcal{F}_{\mu\nu}$ are Lie-algebra valued quantities, i.e. $\mathcal{A}_\mu = A_\mu{}^a T_a$ and $\mathcal{F}_{\mu\nu} = F_{\mu\nu}{}^a T_a$ respectively with $T_a \in \mathfrak{su}(2)$ being the traceless anti-hermitean $N \times N$ generators. The generators satisfy $[T_a, T_b] = f_{abc}T_c$ with real structure constants and $\mathrm{tr}\,T_aT_b = -\frac{1}{2}\delta_{ab}$. For $\mathfrak{su}(2)$ we have $T_a = -\frac{i}{2}\sigma_a$. A *Yang-Mills instanton* is a solution to the equations of motion with finite action. We evalutate the trace in (1) and rewrite the action

   $$\frac{1}{4g^2}\int_M d^4x F_{\mu\nu}{}^a F^{\mu\nu a} \equiv \frac{1}{4g^2}\int_M d^4x F_{\mu\nu}F^{\mu\nu}. \tag{2}$$

   Furthermore we define the *dual field strength* $*\mathcal{F}_{\mu\nu}$ as follows

   $$*\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}. \tag{3}$$

   To find solutions with finite action, we require that $\mathcal{F}$ tends to zero at infinity, i.e. $\mathcal{A}$ becomes a pure gauge at the infinity

   $$\mathcal{A}_\mu = U^{-1}\partial_\mu U \text{ as } x^2 \to \infty \text{ and } U \in SU(2). \tag{4}$$

   One can classify fields with this boundary condition. It is the integral over the first *Pontryagin class* given by

   $$k = -\frac{1}{16\pi^2}\int_M d^4x \,\mathrm{tr}\,\mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu}. \tag{5}$$

   One can show that k takes integer-values.

   (a) Show that for

   $$K_\mu = -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\,\mathcal{A}_\nu\left(\partial_\rho\mathcal{A}_\sigma + \frac{2}{3}\mathcal{A}_\rho\mathcal{A}_\sigma\right). \tag{6}$$

   that $\partial^\mu K_\mu = -\frac{1}{16\pi^2}\,\mathrm{tr}\,\mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu}$. Thus, the integral (5) is reduced to an integral over a three-sphere at the infinity and it counts how many times this sphere covers the gauge group three-sphere $S^3 \cong SU(2)$. Thus, mathematically one looks at the third homotopy group $\pi_3(SU(2)) = \mathbb{Z}$ of the gauge group.

(b) Derive the following inequality by completing the square with $*\mathcal{F}$

$$S^E_{YM} \geq \frac{8\pi^2}{g^2}|k|. \tag{7}$$

When is this inequality saturated?
The field configurations with self and anti-selfdual field strength

$$\mathcal{F}_{\mu\nu} = \pm * \mathcal{F}_{\mu\nu} \tag{8}$$

are called *instanton* and *anti-instanton* respectively. Obviously the action is minimized by these field configurations.

(c) We define

$$\nu = -\frac{1}{24\pi^2}\int_{S^3_\infty} d\theta_1 d\theta_2 d\theta_3 \epsilon^{\mu\nu\rho}\,\mathrm{tr}\, g\partial_\mu g^{-1} g\partial_\nu g^{-1} g\partial_\rho g^{-1}. \tag{9}$$

The θ_i are some angle variables which parametrize S^3_∞ at the infinity. How these are chosen is irrelevant. Show that ν is invariant under continuous deformations. It suffices to show that ν is invariant under infinitesimal deformations. Use that for any Lie group, a general infinitesimal transformation is given by

$$\delta g = g\delta\lambda_a(x)T_a \equiv g\delta T. \tag{10}$$

One can show that $k = \nu$, i.e the number k is invariant under deformations. The invariance under these deformations means that it's homotopy invariant. The relation of instantons and homotopy will be briefly explained in the exercise session.

2. **Anomalies**
Consider a massless fermion coupled to a $U(1)$ gauge boson.

$$\mathcal{L} = \bar{\psi}\mathrm{i}\not{D}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Now perform a local infinitesimal axial transformation:

$$\delta\psi(x) = \mathrm{i}\alpha(x)\gamma_5\psi(x)$$

(a) Find that the Lagrangian transforms as $\delta\mathcal{L} = \alpha(x)\partial_\mu j^\mu_5(x)$ with $j^\mu_5(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$ being the axial current.

(b) Since ψ is massless, show that the axial rotation is a symmetry of the theory. What does this imply for the classical theory?

(c) In the quantum theory we also have to transform the functional measure. A long calculation shows that

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\left\{\mathrm{i}\int \mathrm{d}x\ \alpha(x)\left(\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\right\}.$$

Now you should find the Adler–Bell–Jackiew anomaly:

$$\partial_\mu j^\mu_5 = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,.$$

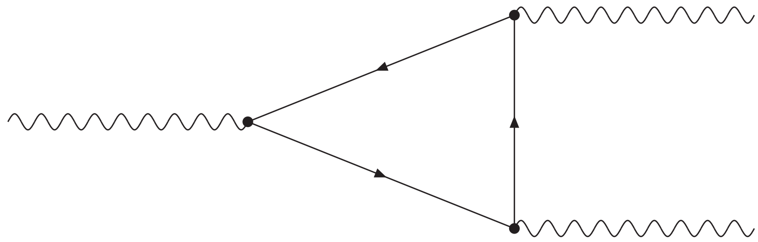

Figure 1: triangle diagram

(d) Show that this leads to a non-conservation of the difference between left- and right-handed fermions in an setup with parallel electric and magnetic fields:

$$\Delta(N_L - N_R) = \frac{e^2}{4\pi^2} \int \mathrm{d}^4x \, \vec{E} \cdot \vec{B}$$

With $N_{L/R} = \int \mathrm{d}^3x \; \bar{\psi}\gamma^0 P_{L/R}\psi$.

(e) This type of anomaly appeares only for the axial current. Now imagine the case of a $U(1)$ gauge theory together with some chiral fermions. Now if they appear in pairs forming a Dirac fermion, then the $U(1)$ couples to the vector current and therefore won't be anomalous. Otherwise, like in the SM, there will usually appear gauge anomalies which break the classical $U(1)$ gauge symmetry on the quantum level. The anomaly can also be explained by the triangle diagrams (see fig. 1) where three gauge bosons couple to a chiral fermion loop. All chiral fermions can appear in the loop, so one has to sum over them with the respective charges as coefficients. Show that in the SM the $SU(2)_L - SU(2)_L - Y$ and the $Y - Y - Y$ anomalies vanish.