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Theoretische Physik

Exercise 9
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SS 09

Exercises on Theoretical Astroparticle Physics

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1. **Boltzmann equation**

Consider a stable species ψ. In a comoving volume, we know that the number of ψ and $\bar{\psi}$ changes only through annihilation and inverse annihilation processes (with χ we indicate all the possible final states):

$$\psi\bar{\psi} \leftrightarrow \chi\bar{\chi}\,. \tag{1}$$

Under certain simplifying assumptions, the Boltzmann equation that rules the evolution of the species ψ can be written as:

$$\frac{dn_\psi}{dt} + 3Hn_\psi = -\langle\sigma_A|v|\rangle \left[n_\psi^2 - (n_\psi^{EQ})^2\right]\,. \tag{2}$$

where $\sigma_A|v|$ is the total annihilation cross section, and n_ψ^{EQ} is the species number density at thermal equilibrium. Let us take a system in which the assumptions that lead to the previous formula are fulfilled, and consider the following questions:

(a) Take a species ψ, and use - as in the lecture - the following quantity

$$Y = \frac{n_\psi}{s}\,, \tag{3}$$

where s is the entropy density. Using the conservation of entropy per comoving volume (sR^3 = constant), show that

$$\dot{n}_\psi + 3Hn_\psi = s\dot{Y}\,. \tag{4}$$

(b) Let m be the mass of the particle ψ. Now introduce the quantity

$$x \equiv \frac{m}{T}\,. \tag{5}$$

During the radiation dominated era, define also $H(m) \equiv 1.67 g_*^{1/2} m^2/m_{Pl}$, and $H(x) = H(m)x^{-2}$. Show that the Boltzmann equation becomes

$$\frac{dY}{dx} = \frac{-x\langle\sigma_A|v|\rangle s}{H(m)}\left(Y^2 - Y_{EQ}^2\right)\,. \tag{6}$$

(c) Write the expression for $Y_{EQ}(x)$ (notice, as a function of x), in the case $x \gg 3$ (that is, the non-relativistic limit), and in the case $3 \gg x$ (the relativistic limit). Suppose that we have freezed out at $x \equiv x_f$ while still in the relativistic case: which is the value of $Y_{EQ}(x)$ at x_f?

(d) We have derived the x-dependent Boltzmann equation

$$\frac{dY}{dx} = \frac{\lambda}{x^2}\left(Y^2 - Y_{EQ}^2\right) , \tag{7}$$

where λ is parametrized by

$$\lambda \equiv \frac{m^3 \langle \sigma_A |v| \rangle}{H(m)} \tag{8}$$

and can be considered as constant in this exercise. At late times, i.e. well after freeze-out, Y will be much larger than Y_{EQ} and the relation

$$\frac{dY}{dx} \simeq \frac{\lambda Y^2}{x^2} \ (x \gg 1) \tag{9}$$

holds. Integrate equation (9) analytically in order to derive the approximation

$$Y_\infty \simeq \frac{x_f}{\lambda} . \tag{10}$$

Typically one can consider Y_f being significantly larger than Y_∞.