Physikalisches Institut
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Theoretische Physik

Exercise 3
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SS 10

Exercises on Group Theory

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–Home Exercises–

H 3.1 Reducibility

(a) Show that a representation D is fully reducible if and only if for every invariant subspace $V_1 \subset V$, $V_1^\perp$ is also an invariant subspace.

(b) Let P denote a projector onto a subspace $V_1 \subset V$. Show that V_1 is an invariant subspace if and only if

$$PD(g)P = D(g)P\,, \qquad \forall g \in G\,.$$

(c) Show that a representation is fully reducible if and only if for every projector P satisfying the equation above also $\mathbb{1} - P$ does. Show that this is equivalent to P and $D(g)$ commuting for all $g \in G$.

H 3.2 Representation of S_3

Consider the three-dimensional representation of S_3 constructed as follows: Choose a basis v_1, v_2, v_3 of $\mathbb{R}^3$. Then $\sigma \in S_3$ acts as

$$D(\sigma):\ v_i \longmapsto v_{\sigma(i)}\,.$$

(a) Find the matrix form of this representation.

(b) Show that the matrix

$$A = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

commutes with all $D(\sigma)$. Using Schur's lemma, what does this imply?

(c) Show that the subspace $V_1 = \langle v_1 + v_2 + v_3 \rangle$ is an invariant subspace.

(d) Show that A is a projector on V_1.

(e) Show that $V_1^\perp$ is also an invariant subspace. *Hint: No calculation!*

(f) Find a basis of $V_1^\perp$. Work out the matrix form of the representation acting on $V_1^\perp$. Is it reducible?

H 3.3 Direct Sums and Tensor Products

Consider two matrices, $A \in \mathbb{K}^{p\times q}$, $B \in \mathbb{K}^{r\times s}$. The direct sum is defined as

$$\begin{aligned} A \oplus B &\in \mathbb{K}^{(p+r)\times(q+s)} \\ (A \oplus B)_{ij} &= \begin{cases} A_{ij} & i \leq p \wedge j \leq q \\ B_{(i-p)(j-q)} & i > p \wedge j > q \\ 0 & \text{else.} \end{cases} \end{aligned}$$

The tensor product is defined as

$$\begin{aligned} A \otimes B &\in \mathbb{K}^{pr\times qs} \\ (A \otimes B)_{(ik)(jl)} &= A_{ij} B_{kl} \,. \end{aligned}$$

In block matrix form they can be visualized as

$$\begin{aligned} A \oplus B &= \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} , \\ A \otimes B &= \begin{pmatrix} A_{11}B & \dots & A_{1q}B \\ \vdots & \ddots & \vdots \\ A_{p1}B & \dots & A_{pq}B \end{pmatrix} . \end{aligned}$$

(a) Show that

$$\begin{aligned} (A \oplus B)^T &= A^T \oplus B^T \,, & (A \oplus B)^* &= A^* \oplus B^* \,, \\ (A \otimes B)^T &= A^T \otimes B^T \,, & (A \otimes B)^* &= A^* \otimes B^* \,. \end{aligned}$$

(b) Show that, if dimensions match,

$$\begin{aligned} (A \oplus B)(C \oplus D) &= AC \oplus BD \,, \\ (A \otimes B)(C \otimes D) &= AC \otimes BD \,. \end{aligned}$$

(c) Let $A \in \mathbb{K}^{m\times m}$, $B \in \mathbb{K}^{n\times n}$. Prove that

$$\begin{aligned} \operatorname{tr} A \oplus B &= \operatorname{tr} A + \operatorname{tr} B \,, & \det A \oplus B &= \det A \cdot \det B \,, \\ \operatorname{tr} A \otimes B &= \operatorname{tr} A \cdot \operatorname{tr} B \,, & \det A \otimes B &= (\det A)^n \cdot (\det B)^m \,. \end{aligned}$$

(d) Given two vecor spaces V, W, each vector in $V \otimes W$ can be represented by a $\dim V \times \dim W$ matrix. Show that the pure vectors $v \otimes w \in V \otimes W$ correspond to rank one matrices.