Physikalisches Institut
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Theoretische Physik

Exercise 4
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SS 10

Exercises on Group Theory

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–Home Exercises–

H 4.1 $SO(3)$ Representation Product

The fundamental, defining and three-dimensional irreducible representation of $SO(3)$ acts on a vector $\phi \in \mathbb{R}^3$ as

$$\phi^i \mapsto R^i_{\ j}\phi^j \,, \qquad R \in SO(3)\,. \tag{1}$$

We take the product representation with itself and denote it by

$$\Phi^{ij} \in \mathbb{R}^3 \otimes \mathbb{R}^3 \cong \mathbb{R}^9\,. \tag{2}$$

(a) What is the representation $\mathscr{R}^{ij}_{\ \ kl}$ transforming Φ^{ij} in terms of $R^i_{\ j}$?

(b) Consider the following operators acting on $\mathbb{R}^9$:

$$(\mathscr{P}_0)^{ij}_{\ \ kl} = \frac{1}{3}\delta^{ij}\delta_{kl}\,, \tag{3}$$

$$(\mathscr{P}_1)^{ij}_{\ \ kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k\right)\,, \tag{4}$$

$$(\mathscr{P}_2)^{ij}_{\ \ kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right) - \frac{1}{3}\delta^{ij}\delta_{kl}\,. \tag{5}$$

Show that they form a complete set of projection operators on $\mathbb{R}^9$, i.e. that $\mathscr{P}_i\mathscr{P}_j = \delta_{ij}\mathscr{P}_i$ and $\sum_i \mathscr{P}_i = \mathbb{1}$.

(c) Show that $[\mathscr{P}_i, \mathscr{R}] = 0$ for $i = 0, 1, 2$.

(d) What does $\mathscr{P}_i \not\propto \mathbb{1}$ then imply using Schur's lemma? What information do you get about the spaces projected on?

(e) How do the matrices $\mathscr{P}_i\Phi$ look like? What is the dimension of $\mathscr{P}_i\mathbb{R}^9$?

At the end you should recover the decomposition

$$|l=1\rangle \otimes |l=1\rangle = |l=0\rangle \oplus |l=1\rangle \oplus |l=2\rangle \tag{6}$$

which you should know from angular momentum addition in quantum mechanics.

H 4.2 Representation of S_3, part 2

Remember last sheet where you constructed a matrix representation of S_3 acting on $\langle v_1 + v_2 + v_3 \rangle^\perp$.

(a) Show the matrix representation you found in H3.2a) is unitary.

(b) Using your basis $\{e_1, e_2\}$ of $\langle v_1 + v_2 + v_3 \rangle^\perp$, compute the matrix $A_{ij} = \langle e_i, e_j \rangle_{\mathbb{R}^3}$ where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ denotes the scalar product in $\mathbb{R}^3$ which makes $\{v_i\}$ an orthobormal basis.

(c) Show that using A as a scalar product, $\hat{D}$ is a unitary representation, i.e. that

$$\hat{D}(\sigma)^T A \hat{D}(\sigma) = A \,, \qquad \text{for all } \sigma \in S_3 \,.$$

H 4.3 Intertwiner

Show that the space of all self-intertwiners of the fundamental representation of $SO(2)$ is isomorphic to $\mathbb{C}$.