Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 12
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SS 10

Exercises on Group Theory

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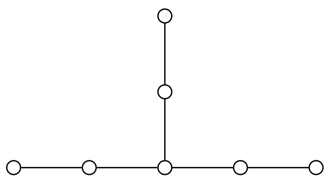
–Bonus Exercises–

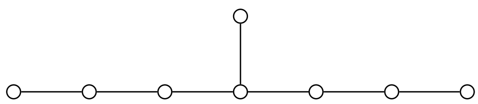
H 12.1 Impossible Π-systems

Show that the following diagrams do not correspond to allowed Π-systems and thus do not represent Lie algebras, by showing that the Cartan matrix has zero determinant or by finding a non trivial linear combination of the simple roots which is a nullvector.

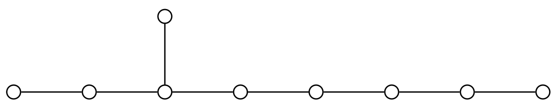
(a)



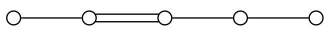
(b)



(c)



(d) In the case of a double line there are two choices which roots are the shorter ones. Show that for the following diagram, none of those choices corresponds to an allowed Π-system. *Hint: You only have to show it for one choice. Why?*



H 12.2 $\boldsymbol{su}(N)$ irreducible representations

(a) Consider the fundamental representation given by the Dynkin label $\Lambda = [1, 0, \ldots, 0] = \frac{2\mu_1\cdot\alpha_i}{\alpha_i\cdot\alpha_i}$. Show by induction that $\omega_k = \mu_1 - \sum_{i=1}^{k-1}\alpha_i$ is also a weight of this representation for $0 < k \leq n$. What is the lowest weight? What is the dimension of the representation?

(b) Show that the weights $\mu_k = \sum_{i=1}^{k}\omega_i$ are the highest weights of the k-fold antisymmetric product of the **N**. What are their dimensions?

(c) Starting with the highest Dynkin label of the k'th fundamental representation,

$$\Lambda_k = [0, \ldots, 0, \underset{\substack{\uparrow \\ k\text{-th}}}{1}, 0, \ldots, 0]$$

show that the lowest Dynkin label of this representation is $-\Lambda_{N-k}$.

(d) For a representation with Dynkin label $\Lambda = [q_1, \ldots, q_{N-1}]$, what is the lowest Dynkin label? What is the highest Dynkin label of the conjugate representation? When is a representation real?

(e) Consider the defining representation **N** and its k-fold tensor product, $V = \bigotimes^k \mathbf{N}$. The vectors can be written as k-index objects, $\psi^{i_1,\cdots,i_k}$, i.e.

$$|\Psi\rangle = \psi^{i_1,\ldots,i_k}|N, i_1\rangle \otimes \ldots \otimes |N, i_k\rangle \in V\,,$$

and we have a natural action of the symmetric group S_k. Show that the symmetrization projectors commute with the SU(N) action, i.e. the possible symmerites of $|\Psi\rangle$ are preserved. *Hint: Why is it sufficient to show that symmetrization and antisymmetrization of two indices commutes with SU(N)? The SU(N) acs as*

$$U|\Psi\rangle = \psi^{i_1,\ldots,i_k}\left(U|N, i_1\rangle\right) \otimes \ldots \otimes \left(U|N, i_k\rangle\right)\,.$$