Exercises on Theoretical Particle Physics II

Prof. Dr. H.P. Nilles – Dr. C. Lüdeling

Due 21.06.2011

9.1 Kaluza Klein reduction of Fermions (*8 credits*)

We want to calculate the spectrum of a Fermion which we dimensionally reduce on a circle. For this, we first investigate in higher dimensional gamma matrices. They satisfy the Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}\,,$$

where we use the convention $\eta^{\mu\nu} = \mathrm{diag}(-,+,\dots,+)$, $\mu,\nu = 0,\dots,D-1$.

(a) Define in addition

$$\Gamma^* = \mathrm{i}^\alpha \Gamma^0 \dots \Gamma^{D-1}\,,$$

where $\alpha \in \mathbb{Z}$. Show that Γ^* anticommutes (commutes) with Γ^μ if D is even (odd). For D odd this implies $\Gamma^* \propto \mathbb{1}$ so from now on D will be even. Determine α from the requirement $\Gamma^*\Gamma^* = \mathbb{1}$. Show that it commutes with the Lorentz group generators

$$\Sigma^{\mu\nu} = \frac{\mathrm{i}}{4}[\Gamma^\mu, \Gamma^\nu]\,,$$

implying that the associated spinor representation is reducible with projectors $P_{L/R} = \frac{1}{2}(\mathbb{1} \pm \Gamma^*)$. (*2 credits*)

(b) Now in $D+1$ dimensions Γ^* plays the role of Γ^D. We start with a $D+1$ dimensional Dirac theory given by

$$\mathcal{L} = \int \mathrm{d}x^{D+1} \bar{\Psi}\Gamma^M \partial_M \Psi + M_{D+1}\bar{\Psi}\Psi\,,$$

where $M = 0,\dots,D$, $\bar{\Psi} = \Psi^\dagger\Gamma^0$. We compactify the theory on a circle of radius R, i.e. $x^D \sim x^D + 2\pi R$. Make an ansatz for Ψ in terms of eigenfunctions of ∂_D with periodic boundary conditions and with Ψ split into $\Psi_{L/R} = P_{L/R}\Psi$. What is the mass matrix for the momentum number modes? Diagonalize it to find the same Kaluza Klein tower as for the Kaluza Klein scalar. (*5 credits*)

(c) Show that for $M_{D+1} = 0$ there are two chiral spinors of opposite chirality in D dimensions. (*1 credit*)

9.2 Gauge Transformation in Kaluza-Klein (*7 credits*)

For a compactification of gravity from $D+1$ to D dimensions we make the ansatz

$$G_{MN} = \phi^\beta \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix}\,, \qquad G^{MN} = \phi^{-\beta} \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & \phi^{-1} + A_\rho A^\rho \end{pmatrix}\,.$$

The $D+1$ dimensional Einstein Hilbert action is

$$S_{D+1} = \frac{1}{16\pi\kappa_{D+1}} \int \mathrm{d}^{D+1}x \sqrt{-G}\, R_{D+1}$$

(a) Determine the exponent β, depending on D, by the requirement that one obtains the D dimensional Einstein Hilbert action after compactification on a circle

$$S_{D+1} \to S_D = \frac{1}{16\pi\kappa_D} \int \mathrm{d}^D x \sqrt{-g}\, R_D + \dots .$$

Hint: Do NOT perform the full dimensional reduction. Just compute $\sqrt{-G}$ *and assume that* $\phi = \text{const.}$ *in* R_{D+1}. *Then* β *follows from cancellation of* ϕ *in front of* R_D. (*4 credits*)

(b) What is the D dimensional Newton constant κ_D in terms of the radius R? (*1 credit*)

(c) Show that $D+1$ diemnsional general coordinate transformations

$$G_{MN} \to \frac{\partial x^R}{\partial x'^M} \frac{\partial x^S}{\partial x'^N} G_{RS}\,,$$

induce D dimensional gauge transformations when reparametrizing the circle coordinate as $x^D \to x^D + \lambda(x^\mu)$, $x^\mu \to x^\mu$. (*2 credits*)

9.3 From 11D SUGRA to 10D Type IIA SUGRA (*5 credits*)

The field content of eleven dimensional supergravity is the metric $g_{MN} = g_{NM}$, an antisymmetric three-tensor A_{MNR} and a gravitino Ψ_M with $\Gamma^M \Psi_M = 0$.

(a) How many degrees of freedom do these fields have? Remember that they transform under the little group. (*1 credit*)

(b) Perform the dimensional reducion to ten dimensions on a circle. What is the massless spectrum, i.e. what representations of the ten dimensional Lorentz group $SO(9,1)$ appear from the given representations of $SO(10,1)$? Count their degrees of freedom and compare to the eleven dimensional case. (*4 credits*)

9.4 Differential forms in three Dimensions (*4 credits*)

Consider a vector field $\vec{v}(x)$ and a scalar field $\phi(x)$ in three Euclidean dimensions. We write the vector as a one form, $v_1(x) = v_i(x)\mathrm{d}x^i$.

(a) Express the known operations div, curl[1], and grad in terms of the exterior derivative d and the Hodge star $*$. (*2 credits*)

(b) Use this to show the identities $\text{div curl} = 0$, $\text{curl grad} = 0$, and $\text{curl curl} = \text{grad div} - \Delta$ with the Laplacian $\Delta = *d*d + d*d*$. (*2 credits*)

[1] also known as rot