Physikalisches Institut
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Theoretische Physik

Exercise 06
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SS 12

# Exercises on String Theory II

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## –Home Exercises–

To be discussed on 14 June 2012

In this exercise sheet we discuss some of the implications of conformal invariance on two dimensional field theories. The first problem is devoted to analyze the properties of the energy momentum tensor. In the second we prove that two point functions are fixed up to a constant factor.

### Exercise 6.1: The Energy Momentum Tensor (*11 credits*)

Consider a theory which is invariant under an infinitesimal transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$. Accoding to Noether's theorem, the current $j_\mu = T_{\mu\nu}\epsilon^\nu$ is a conserved quantity, where

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{1}$$

is the energy momentum tensor.

(a) Show that for theories which are traslationally invariant, $T_{\mu\nu}$ is also a conserved current. (*0.5 credits*)

(b) Prove that conformal invariance of the theory implies the tracelessness of the energy momentum tensor. (*0.5 credits*)

(c) Let us now concentrate on the two dimensional case. Take the cylinder $(\sigma^0, \sigma^1)$ with Minkowski signature and perform the coordinate transformation

$$z' = \sigma^0 + \mathrm{i}\sigma^1\,, \tag{2}$$
$$\bar{z}' = \sigma^0 - \mathrm{i}\sigma^1\,. \tag{3}$$

By setting $z = e^{z'}$, $\bar{z} = e^{\bar{z}'}$, the cylinder is mapped to the complex plane. Compute the metric in terms of these new coordinates. (*2 credits*)

(d) Use your previous results to show that (*3 credits*)

$$\partial_z T_{\bar{z}\bar{z}} = \partial_{\bar{z}} T_{zz} = T_{z\bar{z}} = 0\,. \tag{4}$$

Using these properties, we introduce the notation $T_{zz} = T(z)$, $T_{\bar{z}\bar{z}} = \overline{T}(\bar{z})$.

(e) On the plane, $T(z)$ allows for the expansion

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{l_n}{z^{n+2}} \,. \tag{5}$$

Make use of the OPE

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg} \tag{6}$$

to derive the Virasoro algebra

$$[l_n, l_m] = (m-n)l_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \,. \tag{7}$$

*(5 credits)*

**Exercise 6.2: Two-Point Functions** ***(9 credits)***

Consider the two-point function

$$G(z_1, z_2) = \langle \Phi_1(z_1)\Phi_2(z_2) \rangle \tag{8}$$

for two holomorphic primary fields $\Phi_1$ and $\Phi_2$ with scaling dimensions $\Delta_1$ and $\Delta_2$. Under an infinitesimal conformal tranformation $z \to z + \epsilon$ these fields transform as

$$\delta_\epsilon \Phi_i(z) = [\epsilon(z)\partial + \Delta_i \partial\epsilon(z)]\Phi_i(z) \tag{9}$$

(a) Show that the conformal invariance of the two point function implies

$$[\epsilon(z_1)\partial_1 + \Delta_1\partial\epsilon(z_1) + \epsilon(z_2)\partial_2 + \Delta_2\partial\epsilon(z_2)]G(z_1, z_2) = 0 \,. \tag{10}$$

*(3 credits)*

(b) Set $\epsilon(z) = 1$ in the previous equation and prove that this implies that $G(z_1, z_2)$ depends only on $x = z_1 - z_2$ . *(2 credits)*

(c) Now choose only $\epsilon(z) = z$ to show that $G(x)$ has the form

$$G(x) = \frac{c_{12}}{x^{\Delta_1 + \Delta_2}} \,, \tag{11}$$

with $c_{12}$ a constant. *(2 credits)*

(d) Finally, use the transformation $\epsilon(z) = z^2$ to show that $G(x)$ vanishes unless $\Delta_1 = \Delta_2$ . *(2 credits)*