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Exercise 10
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Exercises on Elementary Particle Physics

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1. *The Renormalization Group and β-Functions*

Consider φ^4 theory in d dimensions, where the Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\,\partial_\mu\varphi_0\,\partial^\mu\varphi_0 - \frac{1}{2}\,m_0^2\,\varphi_0^2 - \frac{1}{4!}\,\lambda_0\,\varphi_0^4 \\ &= \frac{1}{2}\,Z\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\,m_0^2\,Z\,\varphi^2 - \frac{1}{4!}\,\lambda_0\,Z^2\,\varphi^4 + \text{counterterms.}\end{aligned}$$

Let $\Gamma^{(2)}(p,\lambda,m,\mu)$ denote the renormalized *inverse* propagator [cf. Exercise 7, (1.c)].

(a) The propagator is the vacuum expectation value of the time ordered product of 2 fields, i.e.

$$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle.$$

What is the relation between the bare and renormalized inverse propagator?

(b) The bare inverse propagator $\Gamma_0^{(2)}(p,\lambda_0,m_0)$ is independent of the arbitrary mass scale μ introduced by dimensional regularization, i.e.

$$\mu\frac{\partial}{\partial\mu}\Gamma_0^{(2)}(p,\lambda_0,m_0) = 0.$$

Express the bare inverse propagator in terms of the renormalized one and apply the chain rule. Using the definitions

$$\gamma(\lambda) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\log Z, \quad \beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu}, \quad m\,\gamma_m(\lambda) = \mu\frac{\partial m}{\partial\mu},$$

the result can be expressed in a neat form. The result reads

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - 2\gamma(\lambda) + m\,\gamma_m(\lambda)\frac{\partial}{\partial m}\right]\Gamma^{(2)}(p,\lambda,m,\mu) = 0. \tag{1}$$

These equations are called the *renormalization group equations.*

We will now derive a similar equation which describes the behavior of $\Gamma^{(2)}(p,\lambda,m,\mu)$ under a change of scale

$$p \to tp, \quad m \to tm, \quad \mu \to t\mu.$$

The mass dimension of $\Gamma^{(2)}(p,\lambda,m)$ in d dimensions is given by $D = d+2$, i.e.

$$\Gamma^{(2)}(tp,\lambda,m,\mu) = t^D \cdot \Gamma^{(2)}(p,\lambda,t^{-1}m,t^{-1}\mu). \tag{2}$$

(c) Using eq. (2), derive

$$\left[t\frac{\partial}{\partial t} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} - D\right]\Gamma^{(2)}(tp,\lambda,m,\mu) = 0. \tag{3}$$

(d) Using eq. (1), eliminate the term involving μ from eq. (3). The result reads

$$\left[-t\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial \lambda} - n\gamma(\lambda) + m(\gamma_m(\lambda)-1)\frac{\partial}{\partial m} + D\right]\Gamma^{(2)}(tp,\lambda,m,\mu) = 0. \tag{4}$$

Note that if $\beta(\lambda) \equiv \gamma(\lambda) \equiv \gamma_m(\lambda) \equiv 0$, this equation reduces to

$$\left[t\frac{\partial}{\partial t} + m\frac{\partial}{\partial m}\right]\Gamma^{(2)}(tp,\lambda,m) = D \cdot \Gamma^{(2)}(tp,\lambda,m),$$

and the effect of the scaling is simply given by the mass dimension D, as we would naively expect. Renormalization inevitably introduces a different scaling behavior. One way of thinking of eq. (4) is to say that a change in t can be compensated by a change in $\lambda(t)$, $m(t)$, and an overall factor which we will call $f(t)$ such that

$$\Gamma^{(2)}(tp,\lambda,m,\mu) = f(t) \cdot \Gamma^{(2)}(p,\lambda(t),m(t),\mu). \tag{5}$$

This equation relates the inverse propagator at hight momenta tp to the inverse propagator at low momenta p, but with different parameters, $\lambda(t)$ and $m(t)$. This is the so-called "running" of the parameters.

(e) Differentiate eq. (5) with respect to t and derive

$$\left[-t\frac{\partial}{\partial t} + \frac{t}{f}\frac{df}{dt} + t\frac{\partial m}{\partial t}\frac{\partial}{\partial m} + t\frac{\partial \lambda}{\partial t}\frac{\partial}{\partial \lambda}\right]\Gamma^{(2)}(tp,\lambda,m,\mu) = 0. \tag{6}$$

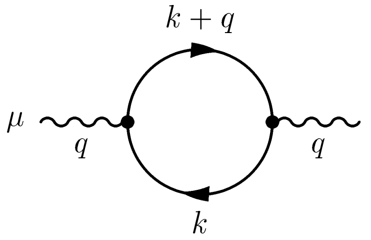
(f) Comparing eqs. (4) and (6) yields

$$\boxed{t\frac{\partial \lambda(t)}{\partial t} = \beta(\lambda), \quad t\frac{\partial m}{\partial t} = m\left[\gamma_m(\lambda) - 1\right], \quad \frac{t}{f}\frac{df}{dt} = D - n\gamma(\lambda).}$$

$\lambda(t)$ and $m(t)$ are *running parameters.* If we know, e.g. $\beta(\lambda)$ (the so-called *β-function*), we can calculate the coupling constant at another energy scale.

2. *Renormalization of the Electric Charge in QED*

   We calculate loop corrections to the photon propagator in QED due to the vacuum polarization diagram. We will see that the correction can be interpreted as a renormalization effect on the electric charge, the QED coupling constant. The vacuum polarization diagram is given by the (amputated) Feynman diagram



   (a) Write down the matrix element $i\Pi^{\mu\nu}(q)$ for this process. Use the QED Feynman rules from 4.1 plus the additional information on loop graphs from 6.1. You will find

$$i\Pi^{\mu\nu}(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \,\mathrm{tr}\left[\gamma^\mu \frac{\not{k}+m}{k^2-m^2+i\epsilon}\gamma^\nu \frac{\not{k}+\not{q}+m}{(k+q)^2-m^2+i\epsilon}\right].$$

   (Hint: The trace comes from the contraction of the 4-spinor indices.)

   (b) Use the trace theorems for γ matrices from 6.1 to simplify the numerator. (Note that the denominator is a scalar with respect to the trace.) The result is

$$4\left\{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k_\rho(k+q)^\rho - m^2)\right\}.$$

   (c) Introduce a Feynman parameter and use the Feynman trick from 6.1 to combine the two denominators. The result is

$$\int_0^1 dx\, \frac{1}{[\ell^2 + x(1-x)\,q^2 - m^2 + i\epsilon]^2},$$

   where $\ell = k + xq$.

   (d) Shift the integration variable from an integration over k to an integration over ℓ and argue that you can drop all terms linear in ℓ.

   (e) Perform a Wick rotation to Euclidean space and substitute $\ell^0 = i\ell^4$. Remember that now you can safely drop the $i\epsilon$ term in the denominator. Thus, we obtain as an intermediate result

$$i\Pi^{\mu\nu}(q) = -4ie^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\frac{1}{2}g^{\mu\nu}\ell^2 - 2x(1-x)\,q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)\,q^2)}{(\ell^2+\Delta)^2},$$

   where $\Delta = m^2 - x(1-x)\,q^2$.

(f) In QED one can prove that, due to the gauge symmetry, all terms proportional to q^μ or q^ν vanish in every S-matrix calculation. Drop the corresponding term from your result. (The proof makes use of the so-called *Ward identity* of QED.)

(g) In Euclidean space we can now change to polar coordinates.

$$\int d^4\ell = \int r^3\, d\Omega_4 = 2\pi^2 \int dr\, r^3.$$

Perform subsequently the substitution $dr \to d(r^2)$.

(h) Let us regularize the divergent integral by a cutoff. Use

$$\int_0^\infty d(r^2) \to \int_0^\Lambda d(r^2)$$

Verify that in the limit of large Λ the following approximations hold

$$\int_0^{\Lambda^2} \frac{x}{[x+\Delta]^2} \to \ln\frac{\Lambda^2}{\Delta} - 1, \qquad \int_0^{\Lambda^2} \frac{x^2}{[x+\Delta]^2} \to \Lambda^2 - 2\Delta\ln\frac{\Lambda^2}{\Delta} + \Delta$$

in order to obtain

$$i\Pi^{\mu\nu}(q) = -\frac{ie^2}{4\pi^2}\int_0^1 dx\, \left\{\frac{1}{2}g^{\mu\nu}\left(\Lambda^2 - 2\,\Delta\ln\frac{\Lambda^2}{\Delta} + \Delta\right) + \right.$$
$$\left. + g^{\mu\nu}\left[x(1-x)\,q^2 + m^2\right]\left(\ln\frac{\Lambda^2}{\Delta} - 1\right)\right\}.$$

(i) This result is not gauge invariant, because the cutoff regularisation does not respect the QED symmetry. We can, however, restore the symmetry by discarding all terms that are not proportional to q^2. (The terms not proportional to q^2 would give rise to a photon mass which is not allowed by the gauge symmetry.)

(j) In the remaining terms, apply our knowledge about the cutoff value $\Lambda \gg q^2$ and $\Lambda \gg m^2$. The final result is

$$i\Pi^{\mu\nu}(q) = \frac{ie^2}{12\pi^2}g^{\mu\nu}q^2\ln\frac{m^2}{\Lambda^2}.$$

(k) We can now use this result to calculate the loop corrected photon propagator. Calculate the correction at one loop and find the result that the propagator is given by

$$-\frac{ig^{\mu\nu}}{q^2}\left[1 + \frac{e^2}{12\pi^2}\ln\frac{m^2}{\Lambda^2}\right].$$

(l) Now calculate the correction to all orders. Using the geometric series as in 7.1 you will obtain

$$-\frac{ig^{\mu\nu}}{q^2}\left[\frac{1}{1-\frac{e^2}{12\pi^2}\ln\frac{m^2}{\Lambda^2}}\right] =: -\frac{ig^{\mu\nu}}{q^2}Z_3.$$

As every propagator ends in two vertices, we can also use our original propagator and multiply $\sqrt{Z_3}$ to each vertex $ie\gamma^\mu$ (see 4.1) instead. Thus, we can regard $\sqrt{Z_3}$ as a factor multiplying the electromagnetic charge which gives the *renormalized charge* or *renormalized coupling constant*:

$$e_R := \sqrt{Z_3}\, e^2$$

Note that it is the renormalized charge that is measured in experiments. In order to distinguish the renormalized (physical) charge from the original parameter e in the Lagrangian, we speak of e as the *bare charge* or *bare coupling constant.*