Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 1
24. Okt. 2005
WS 05/06

Exercises on Elementary Particle Physics

Prof. Dr. H.-P. Nilles

1. *Some Calculations for the Lecture*

   In the lecture, the following equations defined our system of units:

$$\begin{aligned} c &= 2.998 \times 10^8 \ \frac{\mathrm{m}}{\mathrm{s}} = 1 \\ \hbar &= 1.055 \times 10^{-34} \ \mathrm{Js} = 1 \end{aligned}$$

   In particle physics, it is common to measure energies in units of GeV, not in J. The conversion factor is given by:

$$1 \ \mathrm{J} = 6.241 \times 10^9 \ \mathrm{GeV}$$

   (a) Calculate the conversion factors for cm in GeV^{-1} and s in GeV^{-1}, i.e. show that

$$\begin{aligned} 1 \ \mathrm{cm} &\approx 5.07 \times 10^{13} \ \mathrm{GeV}^{-1} \\ 1 \ \mathrm{s} &\approx 1.52 \times 10^{24} \ \mathrm{GeV}^{-1} \ . \end{aligned}$$

2. *The Dirac Equation*

   Using the operator substitutions ($\hbar = 1$)

$$\begin{aligned} \vec{p} &\rightarrow -i\vec{\nabla} \\ E &\rightarrow i\partial_t \end{aligned}$$

   it is possible to get the equations for quantum mechanics from the energy-momentum relations. E.g. from the non-relativistic equation $E = p^2/2m$ one obtains the Schroedinger equation.

   (a) Obtain the Klein-Gordon equation from the relativistic energy-momentum relation ($c = 1$)

$$E^2 = \vec{p}^2 + m^2 \ .$$

   Dirac's basic idea was to "factorize" the Klein-Gordon equation to obtain an equation which is first-order in the derivatives.

(b) Make the ansatz

$$H\psi = (\alpha_i p_i + \beta m)\psi \ . \tag{1}$$

Squaring the Hamilton operator eq. (1) and using $H^2\psi = E^2\psi$ should give the *Klein-Gordon* equation. Show that from this requirement, it follows:

$$\beta^2 = \alpha_i^2 = \mathbb{1} \quad \text{and} \quad \{\beta, \alpha_i\} = \{\alpha_i, \alpha_j\} = 0, \quad i \neq j$$

(c) Why are the α_i and the β not numbers? Why do they have to be hermitian? (A hermitian $\leftrightarrow$ $A^\dagger = A$) What does it imply?

(d) Define the Dirac matrices γ^μ, $\mu = 0, \ldots, 3$ by

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i, \quad i = 1, 2, 3 \ .$$

Show that the Dirac equation $H\psi = E\psi$ can be written in the covariant form

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \ . \tag{2}$$

(e) Show that the gamma matrices fulfill the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1} \ , \qquad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1). \tag{3}$$

(f) Show the following relations:

$$\begin{aligned} \gamma^{0\dagger} &= \gamma^0 & \gamma^{k\dagger} &= -\gamma^k & & \\ \left(\gamma^0\right)^2 &= \mathbb{1} & \left(\gamma^k\right)^2 &= -\mathbb{1} & \gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0 \end{aligned}$$

The lowest dimensional matrices satisfying the Clifforf algebra eq. (3) are 4×4. The choice of the matrices is not unique. We give two representations: the Weyl (or chiral) representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{4}$$

and the Dirac-Pauli representation:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{5}$$

Whereas the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the anticommutation relation $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$.

(g) Verify that each set of matrices eqs. (4, 5) fulfills the Clifford algebra eq. (3).

3. *Free solutions of the Dirac equation*

Since H is represented by a 4×4 matrix, the ψ's are four-component column 'vectors' (called spinors):

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$$

(a) Use the covariant form of the Dirac equation eq. (2) to show that for every ψ_α, $\alpha = 1, \ldots, 4$:

$$(\Box + m^2)\psi_\alpha = 0$$

(Note: $\alpha = 1, \ldots, 4$ has nothing to do with a space-time index $\mu = 0, \ldots, 3$.)

For free particles we can therefore make the following ansatz:

$$\psi = u(p)e^{-ip \cdot x}$$

(b) Plug this ansatz into (1) and use the Dirac-Pauli representation eq. (5) to show that

$$Hu = \begin{pmatrix} m\mathbb{1}_2 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m\mathbb{1}_2 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix} ,$$

with u split into two two-component spinors u_A and u_B.

(c) What are the energy eigenvalues for a particle at rest? Interpret the result.

(d) Now take $\vec{p} \neq 0$. We will label the solutions by an index (s). You can find two solutions by choosing $u_A^{(s)} = \chi^{(s)}$ with $\chi^{(1)} = (1, 0)^T$ and $\chi^{(2)} = (0, 1)^T$. What are the corresponding u_B? What can you say about the energy eigenvalues of this solutions? Proceed analogously for the remaining two solutions. Don't bother about normalizations for now.

(e) It's convenient to choose the so called covariant normalization

$$\int \psi^\dagger \psi dV = 2E \, .$$

Use this to derive the normalizations of the $u^{(s)}$s.

From the solutions, we see that there are always two solutions per eigenvalue and we therefore got a degeneracy. Such degeneracies are always due to additional symmetries.

(f) Show that the operator

$$\Sigma \cdot \hat{p} \equiv \begin{pmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{pmatrix} \qquad \text{with} \quad \hat{\vec{p}} \equiv \vec{p}/|\vec{p}|$$

corresponds to an observable, i.e. that it commutes with H and P. The associated quantum number $\frac{1}{2}\vec{\sigma} \cdot \hat{\vec{p}}$ is called *helicity*. Choose $\vec{p}$ along the z-axis. What are the helicities of the $\chi^{(s)}$?