Universität Bonn
Physikalisches Institut
Theoretische Physik

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Example sheet 10
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General Relativity

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1. *Gravitational waves*

In order to describe gravitational waves, in the lecture the metric was decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{1}$$

with $h_{\mu\nu} \ll 1$, so that we could work in linear order in h. The Einstein field equations in free space in this *weak field approximation* read

$$\partial^\lambda \partial_\lambda h_{\mu\nu} - \partial_\mu \partial_\lambda h^\lambda{}_\nu - \partial_\nu \partial_\lambda h^\lambda{}_\mu + \partial_\mu \partial_\nu h^\lambda{}_\lambda = 0 \ . \tag{2}$$

(a) Show that the field equations in the *harmonic gauge*

$$\partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu h^\mu{}_\mu \tag{3}$$

reduce to a wave equation.

(b) Make an ansatz for the solution of the field equation for gravitational waves

$$h_{\mu\nu}(x) = e_{\mu\nu} \exp(ik^\lambda x_\lambda) + e^*_{\mu\nu} \exp(-ik^\lambda x_\lambda) \ . \tag{4}$$

Show that h solves the field equations if

$$k^\mu k_\mu = 0 \tag{5}$$

and that the choice of a harmonic coordinate system Eq. (3) corresponds to

$$k_\mu e^\mu{}_\nu = \frac{1}{2} k_\nu e^\mu{}_\mu \ . \tag{6}$$

Why is the matrix $e_{\mu\nu}$ symmetric?

(c) Consider a wave traveling in z-direction, i.e.

$$k^1 = k^2 = 0 \ \text{ and } \ k^3 = k^0 =: k > 0 \ . \tag{7}$$

Express e_{i0} $(1 \leq i \leq 3)$ and e_{22} in terms of the other $e_{\mu\nu}$'s.

(d) How does $h_{\mu\nu}$ change under a coordinate transformation

$$x^\mu \longmapsto x'^\mu \equiv x^\mu + \varepsilon^\mu \ ? \tag{8}$$

Perform a coordinate transformation with

$$\varepsilon^\mu(x) = i\epsilon^\mu \exp(ik^\lambda x_\lambda) - i\epsilon^{\mu *} \exp(-ik^\lambda x_\lambda) \tag{9}$$

and determine the $h_{\mu\nu}$.

(e) Invent a coordinate transformation that brings all $e_{\mu\nu}$ to 0 except for e_{11}, e_{12} and e_{22}. How many physical components does h have?

* ***2. Robertson-Walker metric***

The *Robertson-Walker metric* reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t) \left\{ \frac{\mathrm{d}r^2}{1 - \alpha r^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \right\} \tag{10}$$

with $\alpha = 0, \pm 1$. Compute

(a) the Christoffel symbols.

(b) the spatial Riemann tensor, the spatial Ricci tensor and the spatial curvature scalar.

(c) the (4-dimensional) Riemann and Ricci tensors as well as the curvature scalar.

♡ *The exam will be on Thursday, February* 8^{th}*, 2007 from 14:15pm - 16:00pm in the lecture hall I.*