General Relativity and Cosmology

Winter term 2008/09

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Example sheet 10

1. **Motion in Schwarzschild solution**

   Consider the Schwarzschild metric written in the general form (see ex. sheet 5, problem 4)

   $$ds^2 = A(r)\,dt^2 + B(r)\,dr^2 + r^2\,\left[d\theta^2 + \sin^2\theta\,d\phi^2\right] \tag{1}$$

   (a) Write the equations of motion for this metric (keep for the moment A and B general)

   (b) Setting $\theta = \pi/2$ (why can this be done?), integrate suitably to get

   $$\frac{dt}{dp} = \frac{1}{B(r)}\,, \qquad r^2\frac{d\phi}{dp} = J = const. \tag{2}$$

   and

   $$B(r)\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{A(r)} = -E = const. \tag{3}$$

   where p is the parameter along the worldline.

   (c) Show that $d\tau^2 = E\,dp^2$, and therefore $E = 0$ must hold for photons, while $E > 0$ for other matter.

   (d) Eliminate dp from the integrals of motion obtained in b) to get a relation between r and ϕ. Show that

   $$\phi = \pm\int \frac{\sqrt{B(r)}\,dr}{r^2\sqrt{\frac{1}{J^2\,A(r)} - \frac{E}{J^2} - \frac{1}{r^2}}} \tag{4}$$

   is a solution.

2. **Light deflection**

   A photon approaches the central mass from infinity along the direction $\phi_\infty = 0$ with impact parameter b. We want to calculate the deflection of its trajectory. Let r_0 be the radius of the closest approach.

   (a) Determine the value of J in terms of r_0

   (b) Show that now (4) reduces to

   $$\phi(r) = \phi_\infty + \int_r^\infty \frac{\sqrt{B(r')}}{r^2\sqrt{\frac{r'^2}{r_0^2}\frac{A(r_0)}{A(r')} - 1}}\,\frac{dr'}{r'} \tag{5}$$

   (c) Show that the Schwarzschild line element calculated in the class, can be approximated by

   $$B(r) \sim 1 + \frac{2MG}{r}\,, \qquad A(r) \sim 1 - \frac{2MG}{r} \tag{6}$$

   in regions where Newtonian gravity is valid.

(d) Use (5) to calculate the deflection angle $\Delta\phi = 2|\phi(r_0) - \phi_\infty| - \pi$.

Use this equality which holds to lowest order in $2MG/r$ *:*

$$\frac{r^2}{r_0^2}\frac{A(r_0)}{A(r)} - 1 = \left[\frac{r^2}{r_0^2} - 1\right]\left[1 - \frac{2MG\,r}{r_0(r + r_0)}\right] \tag{7}$$

The following integrals may be useful:

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\,\mathrm{arcos}\frac{a}{x}\,, \qquad \int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2\,x}\,, \qquad \int \frac{dx}{(x + a)\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a(x + a)}\,.$$

3. **Precession of perihelia**

Use eqs. (3, 4, 6) to

(a) Determine E and J^2 by looking at the aphelion $r = r_+$ and perihelion $r = r_-$ of a planet in bound orbit around the sun, for general $A(r)$. (*At* $r_\pm$, $dr/d\phi$ *vanishes* .)

(b) Show that the amount of orbital precession per revolution is

$$\Delta\phi = 2|\phi(r_+) - \phi(r_-)| - 2\pi\,, \tag{8}$$

where

$$\phi(r_+) - \phi(r_-) = \int_{r_-}^{r_+} \left[\frac{r_-^2(A^{-1}(r) - A^{-1}(r_-)) - r_+^2(A^{-1}(r) - A^{-1}(r_+))}{r_-^2 r_+^2(A^{-1}(r_+) - A^{-1}(r_-))} - \frac{1}{r^2}\right]^{-1/2} \times \frac{\sqrt{B(r)}\,dr}{r^2} \tag{9}$$

(c) Show that for weak fields, we can use

$$A^{-1}(r) = 1 + \frac{2MG}{r} + \frac{4M^2G^2}{r^2} \tag{10}$$

which makes the first term in (9) quadratic in $1/r$, and that we can then write

$$\phi(r_+) - \phi(r_-) = \int_{r_-}^{r_+} \left[C\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right)\right]^{-1/2} \times \frac{\sqrt{B(r)}\,dr}{r^2} \tag{11}$$

(d) Determine C in the limit $r \to \infty$. you should get

$$C \sim 1 - \frac{4MG}{L} + \cdots\,, \tag{12}$$

where

$$\frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right)\,.$$

Show that (9) now reduces to

$$\phi(r_+) - \phi(r_-) = \left(1 + \frac{2MG}{L}\right) \times \int_{r_-}^{r_+} \frac{\left(1 + \frac{MG}{r}\right)\,dr}{r^2\sqrt{\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right)}} \tag{13}$$

(e) Calculate $\Delta\phi$. (*You can approximate the result of the integral with* $(1 + \frac{MG}{L}\pi)$).

(f) Determine the total precession $\Delta\phi$ for Mercury over the time of a century. (415 revolutions per century; $L = 55.3 \times 10^9$m; $MG = 1475$m). The observed value os 43.11 ± 0.45 arcseconds.