General Relativity and Cosmology

Winter term 2008/09

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Example sheet 12

1. **Penrose diagrams**

   Consider the following spacetime[1]

$$ds^2 = -h^{-1}dt^2 + h\,dr^2 + t^2(d\theta^2 + \sinh^2\theta\,d\phi^2) \tag{1}$$

   where $h = 1 - 2P/t$ $(c = 1)$.

   (a) Describe the asymptotic and horizon structure of this spacetime.

   (b) Follow the coordinate transformations you studied (see Ex. sheet 11) for the Schwarzschild geometry to find the Kruskal diagram for this spacetime.

   (c) Make a final coordinate transformation to get to the final Penrose diagram for this geometry and discuss its structure.

2. **Gravitational waves**

   In order to describe gravitational waves, we decompose the metric into Minkowski metric η and a perturbation h,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2}$$

   h is a small, i. e. $h_{\mu\nu} \ll 1$, so that we can work in linear order in h throughout this problem.

   (a) Consider a coordinate change

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x) \tag{3}$$

   where $\partial\varepsilon^\mu/\partial x^\mu$ is at most of the same order of magnitude as $h_{\mu\nu}$. Calculate the metric in the new coordinate system (described by x').

   (b) Make an ansatz for the solution of the field equations for gravitational waves:

$$h_{\mu\nu}(x) = e_{\mu\nu}\,\exp(i\,k_\lambda x^\lambda) + e^*_{\mu\nu}\,\exp(-i\,k_\lambda x^\lambda) \tag{4}$$

   Show that h solves the field equations (see eq. (9 below)) if

$$k_\lambda k^\lambda = 0 \tag{5}$$

   and that the choice of a harmonic coordinate system (cf. eq.(10)) corresponds to

$$k_\lambda\, e^\lambda_\nu = \frac{1}{2}\, k_\nu\, e^\lambda_\lambda \tag{6}$$

   Why is the matrix $e_{\mu\nu}$ symmetric?

[1]This spacetime can be simply obtained from the Schwarzschild solution by an analytic continuation defined by $r \to it$, $t \to ir$, $\theta \to i\theta$, $\phi \to i\phi$, $m \to iP$ and it is sometimes referred to as *S0-brane*.

(c) Consider a wave traveling in z-direction, i. e.

$$k^1 = k^2 = 0 \qquad \text{and} \qquad k^3 = k^0 =: k > 0 \tag{7}$$

Express e_{i0} ($1 \leq i \leq 3$) and e_{22} in terms of the other $e_{\mu\nu S}$

(d) Perform the coordinate transformation eq. (3) with

$$\varepsilon^\mu(x) = i\,\epsilon^\mu\,\exp(i\,k_\lambda x^\lambda) - i\,\epsilon^{*\mu}\,\exp(-i\,k_\lambda x^\lambda) \tag{8}$$

How does h change?

(e) Invent a coordinate transformation that brings all $e_{\mu\nu}$ to 0 except for e_{11}, e_{12} and e_{22}. How many physical components does h have?

(f) How does h (and $e_{\mu\nu}$) change when we subject the coordinate system to a rotation about the $z-$axis? Discuss the result! How does the situation compare to the case of electromagnetic waves?

Hint: the field equations for free gravitational waves read

$$\Box\, h_{\mu\nu} = 0 \tag{9}$$

One can simplify further the calculation by working with harmonic coordinates, where

$$\frac{\partial}{\partial x^\mu}\, h^\mu{}_\nu = \frac{1}{2}\,\frac{\partial}{\partial x^\nu}\, h^\mu{}_\mu \tag{10}$$