General Relativity and Cosmology

Winter term 2008/09

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Example sheet 6

1. **Lie Derivative**

   (a) Show that
   $$\pounds_u g_{\alpha\beta} = u_{\alpha;\beta} + u_{\beta;\alpha}$$
   by directly computing the difference $[g'_{\alpha\beta}(Q) - g_{\alpha\beta}(Q)]/\varepsilon$ at point Q.
   (Remember that ; denotes covariant differentiation).

   (b) Show that $\pounds_u A^\alpha = A^\alpha{}_{,\beta} u^\beta - u^\alpha{}_{,\beta} A^\beta$ is equivalent to
   $$\pounds_u A^\alpha = A^\alpha{}_{;\beta} u^\beta - u^\alpha{}_{;\beta} A^\beta$$

   (c) Establish the Leibniz rule for Lie derivatives
   $$\pounds_u (A^\alpha \, p_\beta) = (\pounds_u A^\alpha) p_\beta + A^\alpha (\pounds_u p_\beta)$$

2. **Killing Vectors and Symmetries**

   You have seen that the condition for a ξ^α to be a Killing vector is that
   $$\pounds_\xi g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$$

   (a) Verify that, if the metric is independent of some coordinate $x^{\sigma*}$, the vector $\partial_{\sigma*}$ satisfies the Killing equation.

   (b) It is possible to show that in n dimensions, the maximum number of Killing vectors is given by $n(n+1)/2$ (you can try to show this). Consider the 3D metric
   $$ds^2 = dx^2 + dy^2 + dz^2$$
   Find all the Killing vectors associated to this space. What are the isometries associated to these vectors?

   (c) Use the result above to find all Killing vectors of the two-sphere S^2 with metric
   $$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

   (d) Show that the three vectors you found above, satisfy the algebra
   $$[R, S] = T \qquad [S, T] = R\,, \qquad [T, R] = S\,.$$