General Relativity and Cosmology

Winter term 2008/09

Dr. S. Förste

Example sheet 7

1. **Curvature**

   (a) The covariant derivative of a tensor in a certain direction measures how much the tensor changes relative to what it would have been if it had been parallel transported, since the covariant derivative of a tensor in a direction along which it is parallel transported is zero. The commutator of two covariant derivatives, then, measures the difference between parallel transporting the tensor first on one way, and then on the other, versus the opposite ordering. Consider a vector field V^ρ, show that the commutator of two covariant derivatives on this field is given by

   $$[\nabla_\mu, \nabla_\nu]\, V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} \nabla_\lambda V^\rho \tag{1}$$

   where

   $$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \hat{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \hat{\Gamma}^\rho_{\mu\sigma} + \hat{\Gamma}^\rho_{\mu\lambda} \hat{\Gamma}^\lambda_{\nu\sigma} - \hat{\Gamma}^\rho_{\nu\lambda} \hat{\Gamma}^\lambda_{\mu\sigma} \tag{2}$$

   is the *curvature Riemann tensor* and

   $$T^\lambda_{\mu\nu} = 2\hat{\Gamma}^\lambda_{[\mu\nu]} \tag{3}$$

   is the *torsion tensor* you have seen before. Notice from this derivation that the expression (2) is constructed from nontensorial elements. However, you can check (homework) that the transformation laws all work out to make this particular combination a legitimate tensor. Notice also that the antisymmetry of $R^\rho{}_{\sigma\mu\nu}$ in the last two indices is immediate from (2) and its derivation. Finally, note that the curvature tensor above derived was constructed from the connection (no mention of the metric was made). Thus this expression is true for any connection, whether or not it is metric compatible or torsion free.

   (b) Use locally inertial coordinates to deduce the symmetry properties of the curvature tensor. This is most easily done using the Riemann tensor with all lower indices $R_{\rho\sigma\mu\nu} = g_{\sigma\lambda} R^\lambda_{\sigma\mu\nu}$ (remember that tensorial equations are true in any coordinate system).

   (c) Show that the sum of cyclic permutations of the last three indices of the curvature tensor vanishes, that is

   $$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \tag{4}$$

   (d) Use the results in (b) to show that (4) is equivalent to the vanishing of the antisymmetric part of the last three indices of the Riemann tensor, that is:

   $$R_{\rho[\sigma\mu\nu]} = 0 \tag{5}$$

   (e) Given these relationships between the different components of the Riemann tensor, how many independent quantities remain? Deduce the number of independent components of the Riemann tensor in n dimensions.

(f) Make use once more of locally inertial coordinates to prove *Bianchi's identity* of the Riemann tensor:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \tag{6}$$

(g) By contracting eq. (6) twice, show that

$$\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R$$

Use this to prove that the Einstein tensor satisfies

$$\nabla^{\mu} G_{\mu\nu} = 0$$

(h) Show that any Killing vector ξ^{μ} satisfies

$$\nabla_{\mu}\nabla_{\sigma}\xi^{\rho} = R^{\rho}{}_{\sigma\mu\nu}\xi^{\nu}$$

(i) **(For the brave).** The Weyl tensor in n dimensions is given by

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma}R$$

One of the most important properties of this tensor is that it is invariant under *conformal transformations*. This means that $C^{\rho}{}_{\sigma\mu\nu}$ (note that the first index is upstairs) computed for some metric $g_{\mu\nu}$ is the same as that computed for the *conformal metric* given by $\omega^2(x)g_{\mu\nu}$, where $\omega(x)$ is a nonvanishing arbitrary function of spacetime. Show that the Weyl tensor is invariant by a conformal transformation.

2. **Curvature: hands on**

(a) Consider again the two sphere S^2 with metric

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Compute all the components of the Riemann tensor.

(b) Consider now the three sphere in coordinates $x^{\mu} = (\psi, \theta, \phi)$ with metric

$$ds^2 = d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Compute all Christoffel connection coefficients, Riemann tensor, Ricci tensor and Ricci scalar.