General Relativity and Cosmology

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Example sheet 8

1. **Useful relations**

   For any matrix A, the exponential e^A is defined by the power series

   $$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

   (a) Show that

   $$\det e^A = e^{\mathrm{tr} A}$$

   (b) Show that under variation $A \to A + \delta A$, the determinant of A varies as

   $$\delta(\det A) = \det A \, \mathrm{tr}(A^{-1}\delta A)$$

   (assume that A is invertible).

   (c) Show that

   $$\frac{\delta}{\delta g^{\mu\nu}} = -g_{\mu\rho}\, g_{\nu\lambda} \frac{\delta}{\delta g_{\rho\lambda}}$$

2. **Physics in curved spacetime**

   (a) Consider the action for a scalar field ϕ in curved spacetime

   $$S = -\int d^4x \sqrt{-g} \left(\frac{1}{2} (\partial\phi)^2 + V(\phi) \right)$$

   where $(\partial\phi)^2 \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ (you will see this notation often!), and $V(\phi)$ is the potential energy for ϕ (e. g. $V(\phi) = \frac{1}{2}m^2\phi^2$). Derive the equation of motion for ϕ by varying the action with respect to the scalar field. How do things change if you consider the same action in n-dimensions, rather than 4?

   (b) Compute the energy momentum tensor for the scalar field, by varying the action above with respect to the metric.

   (c) Consider the Lagrange density for an electromagnetic field in curved space is given by

   $$\mathcal{L} = -\frac{1}{4}\sqrt{-g}\, F^2$$

   where $F^2 \equiv F^{\mu\nu}F_{\mu\nu}$ (again, you will encounter this notation often) and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Derive the equation of motion for F by varying the action with respect to A, and the energy momentum tensor by varying it with respect to the metric.