# String Theory

Winter Term 2008/2009

Problem Sheet 3

Discussion: November 5, 14:00 in Hörsaal 118, AVZ

1. Show that the delta function on the circle is given by

$$\delta(\varphi - \varphi') = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} e^{2\mathrm{i}n(\varphi-\varphi')} ,$$

   where $\varphi$ is taken to range from zero to $\pi$!

2. Show that $X^\mu(\sigma,\tau) = X_L^\mu(\sigma_+) + X_R^\mu(\sigma_-)$ with

$$X_L^\mu(\sigma_+) = \frac{1}{2}x^\mu + \frac{1}{2}l_{\mathrm{S}}^2\, p^\mu \sigma_+ + \frac{\mathrm{i}}{2} l_{\mathrm{S}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2\mathrm{i}n\sigma_+}$$

$$X_R^\mu(\sigma_-) = \frac{1}{2}x^\mu + \frac{1}{2}l_{\mathrm{S}}^2\, p^\mu \sigma_- + \frac{\mathrm{i}}{2} l_{\mathrm{S}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-2\mathrm{i}n\sigma_-}$$

   is indeed the general solution to $\partial_+\partial_- X^\mu = 0$ and the boundary conditions $X^\mu(\sigma+\pi,\tau) = X^\mu(\sigma,\tau)$!

3. Find the mode expansion for

   (a) twisted closed strings, which are defined by the boundary condition

$$X^\mu(\sigma+\pi,\tau) = -X^\mu(\sigma,\tau)$$

   (b) Neumann–Dirichlet open strings with boundary conditions

$$X^\mu(0,\tau) = 0 \qquad \text{Dirichlet at } \sigma = 0$$

$$\partial_\sigma X^\mu(\sigma,\tau)\,\big|_{\sigma=\pi} = 0 \qquad \text{Neumann at } \sigma = \pi$$

4. (a) Verify that the canonical Poisson brackets

$$[P^\mu(\sigma,\tau)\,, X^\nu\,(\sigma',\tau)]_{\mathrm{PB}} = \eta^{\mu\nu}\delta(\sigma-\sigma')\ , \qquad [P^\mu, P^\nu]_{\mathrm{PB}} = [X^\mu, X^\nu]_{\mathrm{PB}} = 0$$

   lead to the algebra of the $\alpha$'s,

$$[\alpha_m^\mu, \alpha_n^\nu]_{\mathrm{PB}} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{\mathrm{PB}} = \mathrm{i}m\eta^{\mu\nu}\delta_{m+n,0}\,, \qquad [\alpha_m^\mu, \tilde{\alpha}_n^\nu]_{\mathrm{PB}} = 0\,!$$

(b) Derive the Virasoro algebra

$$[L_m, L_n]_{\text{PB}} = \mathrm{i}\,(m-n)\,L_{m+n}\,,$$

where the generators $L_m$ are defined as

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n\,.$$

Note that the Poisson brackets are still classical, so you don't have to worry about operator ordering!

5. Compute the central charge (or anomaly) term $A(m)$ in the quantised Virasoro algebra,

$$[L_m, L_n] = (m-n)\,L_{m+n} + A(m)\,\delta_{m+n,0}\,.$$

(a) Why does this term only arise for $m+n=0$?

(b) Using the Jacobi identity for the $L_m$, show the recursion relation

$$A(m+1) = \frac{m+2}{m-1}A(m) - \frac{2m+1}{m-1}A(1)\,.$$

(c) Show that this relation implies that $A(m) = cm + dm^3$! Here, $c$ and $d$ are some real coefficients. You can assume that $A(m)$ is polynomial. Exploit the linearity of the recursion relation!

(d) Determine $c$ and $d$ by evaluating the expectation value

$$\langle 0|\,[L_m, L_{-m}]\,|0\rangle$$

in a zero momentum ground state $|0\rangle$ for some suitable values of $m$.

6. Show, along the lines of the lecture, that $D = 26$ is part of the boundary of the ghost-free subspace of the Hilbert space.

Physical spurious (and hence null) states appear for $a-1$, as shown in the lecture by considering spurious states of the form $|\psi\rangle = L{-}1\,|\chi_1\rangle$. Now consider a different state defined as

$$|\psi\rangle = \left(L_{-2} + \gamma L_{-1}^2\right)|\chi\rangle\,,$$

where $\gamma$ is some number.

(a) What conditions must the state $|\chi\rangle$ satisfy in order for $|\psi\rangle$ to be spurious?

(b) Show that requiring $|\psi\rangle$ to be physical, i.e. imposing the Virasoro constraints, leads to $\gamma = \frac{3}{2}$ and $D = 26$.

(c) Consider the excited states $|\xi\rangle = \xi\alpha_{-1}\,|0;k\rangle$ obtained by acting on the momentum-$k$ ground state. These states are labelled by the polarisation vector $\xi_\mu$.

Find the constraints on $\xi_\mu$ obtained from the Virasoro conditions $(L_0 - a)\,|\xi\rangle = L_m\,|\xi\rangle = 0$ for $m > 0$.

How many states are allowed by the constraints? Depending on $a$, what are their masses and norm?