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Theoretische Physik

Exercise 11
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Exercises on Theoretical Particle Physics

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–Class Exercises–

C 11.1 Two dimensional solitons

Consider a two dimensional real Klein Gordon field described by the following action

$$S = \int \mathrm{d}^2x \left\{ \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - \frac{\lambda}{2}\left(\phi^2 - a^2\right)^2 \right\}$$

with $\lambda, a^2 > 0$. The simplest ground state is described by $\phi = \pm a$. It has zero energy and breaks the symmetry $\phi \to -\phi$. But there is another ground state of this theory which we want to investigate now.

(a) We want to find a time independent solution. Write down the time independent Euler Lagrange equation.

(b) Since we want a solution with finite energy, we require that $\phi(x \to \pm\infty) \to \pm a$. But for a non-trivial solution we also require $\phi(x) \neq \pm a$. Integrate the equation using these conditions. *Hint: Remember how you integrated the e.o.m. in the Kepler problem.* $\int 1/(x^2 - a^2)\, dx = 1/a \ \mathrm{Artanh}(x/a)$.

(c) Calculate the energy density and show that it is peaked. Integrate it to find that the mass is indeed finite. How does it behave when you vary the coupling λ while keeping the Lagrangian mass $\mu = a\sqrt{\lambda}$ constant?
Hint: $\int \cosh^{-4}(x)\, dx = 2/3 \tanh(x) + 1/3 \tanh(x) \cosh^{-2}(x)$.

(d) This field configuration has clearly more energy than the constant vacuum so we want to make sure that it does not decay into it. For this we define the current

$$j^\mu(x) = \epsilon^{\mu\nu}\partial_\nu\phi\,.$$

Show that this current is conserved. What is the associated charge Q? Identify the vacua with $Q = 0, \pm 2a$.

Remark: By adding two trivially behaving spatial dimensions we find the description of a domain wall in four dimensions.

C 11.2 't Hooft–Polyakov monopole

Now we want to look at topological field configurations in four dimensions. Again we search for time independent solutions which are stabilized by their behaviour when approaching spatial infinities. Here $\partial\mathbb{R}^3 \cong S^2$.

(a) Argue that one real scalar or complex field with potential of the form $V(\phi) \propto \left(|\phi|^2 - a^2\right)^2$ cannot lead to topologically non-trivial solutions. *Hint: Study the maps $\partial\mathbb{R}^3 \to M$ where M is the set of stable extrema of $V(\phi)$.*

(b) Show that for a pure scalar field theory (of any dimension) there are no topologically non-trivial solutions with finite energy. *Hint: Write down the Hamitonian.* $(\nabla\phi) = (\partial\phi/\partial r)^2 + (\hat{r} \times \nabla\phi)^2$. *How do the terms behave as $r \to \infty$?*

This sicknes can be cured by introducing a gauge symmetry. Then the gauge field in the covariant derivative will cancel the diverging behaviour. Let us investigate this with the so-called *'t Hooft Polyakov model.* Here we have an $SO(3)$ gauge field and a triplett of real scalars. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu,a}F_{\mu\nu,a} + \frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)^a - \frac{\lambda}{4}\left(\phi_a\phi^a - a^2\right)^2 ,$$

with

$$\begin{aligned} F_{\mu\nu,a} &= \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} - e\epsilon_{abc}A_{\mu,b}A_{\nu,c} , \\ \left(D_\mu\phi\right)_a &= \partial_\mu\phi_a - e\epsilon_{abc}A_{\mu,b}\phi_c . \end{aligned}$$

(c) Describe the set of minima of the scalar potential. How does the topologically trivial vacuum look? What is the unbroken gauge symmetry?

(d) Argue that $\phi_a(r \to \infty) \to a\hat{r}_a$ is topologically stable. Describe this configuration geometrically.

(e) Show that if the gauge field behaves like $A_{0,a} = 0$, $A_{i,a}(r \to \infty) \to \epsilon_{iab}\frac{r_b}{er^2} + \mathcal{O}(r^{-2})$, the dangerous terms in the energy density cancel.

(f) Show that this field configuration corresponds to a magnetic monopole of charge $g = 4\pi/e$. *Hint: The surviving U(1) generator and field strength is the one projected on the scalar fields vev.*