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## Exercises on Theoretical Particle Physics

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–HOME EXERCISES–  
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### H 5.1 Gell-Mann Matrices

*0,5+4+2+1,5=8 points*

The standard basis for the fundamental representation of  $\mathfrak{su}(3)$  is

$$T^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$T^5 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^7 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

- (a) Why are there exactly 8 matrices in the basis?
- (b) Evaluate the commutators of these matrices to determine the structure constants  $f^{abc}$ . Show that, with the normalizations used here,  $f^{abc}$  is totally antisymmetric. *Hint: This exercise is tedious; you may wish to check only a representative sample of the commutators.*
- (c) Check the orthogonality relation,

$$\mathrm{tr}\{T^a T^b\} = \ell(r)\delta^{ab},$$

where  $\ell(r)$  is a constant for each representation  $r$  and it is known as the **Dynkin index**. Evaluate the constant  $\ell(r)$  for this representation. *Hint: Just check it for a representative sample. What is the trace of a product of a diagonal and a off-diagonal matrix?*

- (d) Why is  $\{T^3, T^8\}$  a good choice for the Cartan subalgebra? Show that it is diagonalized by the (complex) basis transformation,

$$T_{\pm} = T^1 \pm iT^2, \quad U_{\pm} = T^4 \pm iT^5, \quad V_{\pm} = T^6 \pm iT^7.$$

## H 5.2 The Standard Model Higgs effect

$1+2+2+1.5+2+1.5+1+1 = 12$  points

The Glashow–Weinberg–Salam theory is the part of the Standard Model (SM) of particle physics which describes the electroweak interactions by a non-Abelian gauge theory with the gauge group  $SU(2)_L \times U(1)_Y$ . In a one-family approximation, the SM has the following particle content:

	$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$R = e_R$	$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$T^a W_\mu^a$	$B_\mu$
Hypercharge $Y$	-1	-2	+1	0	0
$SU(2)_L$ rep.	<b>2</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>1</b>
Lorentz rep.	(1/2, 0)	(0, 1/2)	(0, 0)	(1/2, 1/2)	(1/2, 1/2)

where  $L$ ,  $R$  contain Dirac spinors and the superscripts in the Higgs doublet denote electromagnetic charges. The corresponding Lagrangian is given by

$$\begin{aligned}
 \mathcal{L} = & \overbrace{\bar{R}(i\gamma^\mu D_\mu)R + \bar{L}(i\gamma^\mu D_\mu)L}^{\text{kinetic energy terms of leptons and interactions with gauge bosons}} \quad \overbrace{-\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}}^{\text{kinetic energy terms of the gauge bosons and self-interactions}} \\
 & \underbrace{+(D_\mu\Phi)^\dagger(D^\mu\Phi) - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2}_{\text{Higgs field with potential}} \quad \underbrace{-G_e(\bar{L}\Phi R + \bar{R}\Phi^\dagger L)}_{\text{electron-Higgs Yukawa coupling}}, \quad (1)
 \end{aligned}$$

with

$$D_\mu = \partial_\mu + ig' \frac{Y}{2} B_\mu + ig T^a W_\mu^a, \quad (2)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c. \quad (3)$$

- Write down how the covariant derivative eq. (2) acts on the left- and right-handed leptons doublets/singlet and on the Higgs-doublet.
- Show that the Lagrangian eq. (1) is Lorentz invariant.
- Show that eq. (1) is gauge invariant as well.
- For the Higgs mechanism to work we need  $\mu^2 < 0$ . For which value of  $|\Phi|$  does the Higgs potential obtain a minimum? By an  $SU(2)_L$  rotation we can choose the vacuum expectation value (VEV) of the Higgs field to be of the form  $\langle \Phi \rangle = \frac{1}{\sqrt{2}}(0, v)^T$ . This leads to a redefinition of the excitation modes of the Higgs fields,

$$\Phi(x) = \exp\left\{\frac{i}{v}\xi^a(x)T^a\right\} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix},$$

with  $\xi^a(x)$  and  $\eta(x)$  being real fields and  $T^a$  the generators of  $SU(2)$ . Now we apply an  $SU(2)_L$  gauge transformation such that the angular excitations  $\xi^a(x)$  vanish. This gauge transformation is called *unitary gauge*. Show that the Higgs potential in the unitary gauge is given by

$$V(\Phi) = -\mu^2\eta^2(x) + \lambda v\eta^3(x) + \frac{\lambda}{4}\eta^4(x).$$

What is the mass of the  $\eta$  field? Compare the degrees of freedom (DOF) in the Higgs sector to the situation before symmetry breakdown.

- (e) Consider the kinetic energy terms of the Higgs field in eq. (1). Show that

$$(D_\mu \Phi)^\dagger (D^\mu \Phi) = \frac{1}{2} \partial_\mu \eta \partial^\mu \eta + \frac{1}{4} g^2 (v + \eta)^2 W_\mu^- W^{+\mu} + \frac{1}{8} (v + \eta)^2 \begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix}, \quad (4)$$

with  $W^{\pm\mu} := \frac{1}{\sqrt{2}}(W^{1\mu} \mp iW^{2\mu})$ .

- (f) The masses of the gauge bosons are given by the terms that are quadratic in the fields, e. g.  $\frac{1}{4}g^2v^2W_\mu^-W^{+\mu} = m_W^2W_\mu^-W^{+\mu}$ , where  $m_W = \frac{1}{2}vg$ . However, to see the masses of  $W_\mu^3$  and  $B_\mu$  one has to diagonalize the matrix in eq. (4):

$$\frac{v^2}{8} \begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix} \mathcal{O}^T \mathcal{O} \begin{pmatrix} g^2 & -g'g \\ -g'g & g'^2 \end{pmatrix} \mathcal{O}^T \mathcal{O} \begin{pmatrix} W^{3\mu} \\ B^\mu \end{pmatrix} = \begin{pmatrix} Z_\mu & A_\mu \end{pmatrix} \begin{pmatrix} m_Z^2 & 0 \\ 0 & m_A^2 \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}.$$

Determine this orthogonal matrix  $\mathcal{O}$  by computing the corresponding eigenvalues and eigenvectors. What are the masses of the  $Z_\mu$  and  $A_\mu$  fields? Compare the DOF in the gauge sector to the situation before the symmetry breakdown. What can you say about the total amount of DOF?

- (g) As you know, an orthogonal  $2 \times 2$  matrix can be written as

$$\mathcal{O} = \begin{pmatrix} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix}.$$

Write  $\cos \theta_W$  in terms of  $g'$  and  $g$ . Show for the ratio of the  $W$ - and  $Z$ -boson masses

$$\frac{m_W}{m_Z} = \cos \theta_W.$$

The angle  $\theta_W$  is sometimes called *Weinberg angle* or *weak mixing angle*.

- (h) Finally, consider the covariant derivative eq. (2). Substitute the fields  $B_\mu$  and  $W_\mu^a$  by  $W_\mu^\pm$ ,  $Z_\mu$  and  $A_\mu$  and show

$$D_\mu = \partial_\mu + iA_\mu e Q + iZ_\mu \frac{1}{\sqrt{g'^2 + g^2}} \left( g^2 T_3 - g'^2 \frac{Y}{2} \right) + \frac{ig}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix},$$

where we have defined the electric charge  $e = \frac{g'g}{\sqrt{g'^2 + g^2}}$  and  $Q := T_3 + \frac{Y}{2}$ .