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## Exercises on Theoretical Particle Physics

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–HOME EXERCISES–  
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### H 11.1 Group-theoretical GUT breaking

$3+3+3+3 = 12$  points

We believe that the SM gauge group unifies to one simple Lie algebra (e. g.  $\mathfrak{su}(5)$ ) which is broken at very high energies  $\mathcal{O}(10^{16}$  GeV). Representations of such Grand Unified Theory (GUT) group decompose into those of the SM gauge group. Hence, tools for this group-theoretical symmetry breaking have to be applied.

**Dynkin's Symmetry Breaking:** To each simple root one assigns an integer number, called the **Kač-label**  $a_i$ . They are given as the coefficients of the decomposition of the highest root in the basis of simple roots. Deleting any node with Kac-label  $a_i = 1$  from the Dynkin diagram gives a maximal regular subalgebra times a  $U(1)$  factor.

- (a) In the case of  $SU(5)$ , all Kač-labels are 1. Apply Dynkin's rule to find the symmetry breaking yielding the SM gauge group, i. e.

$$SU(5) \rightarrow SU(3) \times SU(2) \times U(1).$$

The  $U(1)$  generator is constructed as a Cartan element of  $SU(5)$  such that it is annihilated by all roots of  $SU(3) \times SU(2)$ . Show that  $Q = \text{diag}(-2, -2, -2, 3, 3)$  fulfills these conditions.

- (b) The  $\mathbf{5}$  is a reducible representation of the subgroup  $SU(3) \times SU(2) \times U(1)$ . Let  $\alpha_1$  and  $\alpha_2$  correspond to  $SU(3)$  and  $\alpha_4$  to  $SU(2)$ . Thus, every weight  $\lambda$  of  $SU(5)$  decomposes as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \rightarrow (\lambda_1, \lambda_2 | \lambda_4) = (\mu | \nu).$$

First, write down all weights  $(\mu | \nu)$ , then find the highest weight  $\mu$  and determine all weights and the dimension of the corresponding representation. Consider now the values of  $\nu$  belonging to this  $\mu$ -representation and state the dimension of the  $\nu$ -representation! Repeat these steps starting with the highest weight  $\nu$ . Finally, determine the  $U(1)$  charge by applying the  $U(1)$  generator to the weight vectors. The result reads

$$\mathbf{5} \rightarrow (\mathbf{3}, \mathbf{1})_{-2} \oplus (\mathbf{1}, \mathbf{2})_3.$$

- (c) Repeat the analysis for the representation **10** and verify

$$\mathbf{10} \rightarrow (\mathbf{1}, \mathbf{1})_6 \oplus (\overline{\mathbf{3}}, \mathbf{1})_{-4} \oplus (\mathbf{3}, \mathbf{2})_1.$$

*Hint: All weights which appear in the calculation have multiplicity 1.*

- (d) Perform the breaking for the representation corresponding to the highest weight with Dynkin coefficients  $(1, 0, 0, 1)$ , i. e. the adjoint **24**. The result reads

$$\mathbf{24} \rightarrow (\mathbf{8}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\overline{\mathbf{3}}, \mathbf{2})_5 \oplus (\mathbf{3}, \mathbf{2})_{-5}.$$

Identify the gauge bosons of the standard model.

*Hint: All weights which appear in the calculation have multiplicity 1, except for  $(0, 0, 0, 0)$  in **24** of  $SU(5)$  with multiplicity 4 and  $(0, 0)$  in **8** of  $SU(3)$  with multiplicity 2. What is the origin of this?*

After a renormalization of the  $U(1)$  generator to  $Q' = \frac{1}{6}Q$  we recover one family of the standard model in  $\overline{\mathbf{5}} \oplus \mathbf{10}$ .

## H 11.2 Dynamical GUT breaking

$2+2+2+2 = 8$  points

It is necessary to generalize the Higgs mechanism of the SM to understand the symmetry breaking of any GUT theory to the SM. Thus, we describe the Higgs mechanism for a field  $H$  in an arbitrary representation  $\rho$  of a semi-simple Lie algebra  $\mathfrak{g}$ .

- (a) Consider a complex scalar  $H$  in the representation  $\rho$  of a gauge group  $\mathcal{G}$ . Assume further that  $H$  acquires a vev  $\langle H \rangle$  due to some potential. Deduce from the kinetic term

$$(D_\mu H)^* (D^\mu H)|_1 = (\partial_\mu H + ig\rho(T^a)A_\mu^a H)^* (\partial^\mu H + ig\rho(T^b)A^{b\mu} H)|_1,$$

that a gauge boson  $A_\mu^a$  is massless, if  $\rho(T^a)\langle H \rangle = 0$ . Then,  $T^a$  belongs to the unbroken gauge group  $\mathcal{G}'$ .

Specialize to  $H$  in the adjoint representation with the kinetic term  $\text{Tr}(D_\mu H)^\dagger (D^\mu H)$  and deduce

$$T^a \in \mathcal{G}' \quad \text{if} \quad [T^a, \langle H \rangle] = 0, \quad T^a \notin \mathcal{G}' \quad \text{if} \quad [T^a, \langle H \rangle] \neq 0.$$

Let us apply this for the desired symmetry breaking by introducing a Higgs field in the adjoint of  $SU(5)$ , i. e. a  $5 \times 5$  Hermitian traceless matrix.<sup>1</sup> We work with a scalar potential invariant under  $H \rightarrow -H$  of the form

$$V(H) = -m^2 \text{Tr}(H^2) + \lambda_1 (\text{Tr}(H^2))^2 + \lambda_2 \text{Tr}(H^4).$$

- (b) First, use the previous results to argue that a Higgs field  $H$  precisely in the adjoint **24** is an appropriate choice to break  $SU(5)$  to the SM. Which component of **24** should

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<sup>1</sup>Note that this is not the SM-Higgs field, which is contained in the **5**.

develop the VEV? (cf. exercise H 9.1 (d)) Use the gauge symmetry  $H \rightarrow H' = UHU^\dagger$  to obtain

$$H = \text{diag}(h_1, h_2, h_3, h_4, h_5)$$

and check that the minimum of the potential is given by the same equation  $\forall h_i$ :

$$4\lambda_2 h_i^3 + 4\lambda_1 a h_i - 2m^2 h_i - \mu = 0 \quad \text{with} \quad a = \sum_j h_j^2, \quad \forall i = 1, \dots, 5. \quad (1)$$

Here  $\mu$  is a Lagrange multiplier necessary to impose the constraint  $\sum_i h_i = 0$ .

The cubic equation (1) has at most three roots denoted by  $\phi_1, \phi_2, \phi_3$ . Thus, there are at most three different eigenvalues  $h_i \in \{\phi_1, \phi_2, \phi_3\}$ . Let  $n_i$  be the multiplicity of the eigenvalue  $\phi_i$ ,  $i = 1, 2, 3$ , in  $\langle H \rangle$ :

$$\langle H \rangle := \text{diag}(\phi_{i_1}, \dots, \phi_{i_5}) \quad \text{with} \quad n_1 \phi_1 + n_2 \phi_2 + n_3 \phi_3 = 0.$$

- (c) Following part (a), what is the most general symmetry breaking of  $SU(5)$ ? What happens to the rank of the gauge group? Consider also possible  $U(1)$  factors. Depending on the relative magnitude of the parameters  $\lambda_1$  and  $\lambda_2$ , the combinations  $(3, 2, 0)$ ,  $(2, 2, 1)$  or  $(4, 1, 0)$  for  $(n_1, n_2, n_3)$  minimize the potential. Thus,

$$\text{case1: } SU(5) \rightarrow SU(3) \times SU(2) \times U(1), \quad \text{case2: } SU(5) \rightarrow SU(4) \times U(1), \quad (2)$$

which gives restrictions on phenomenologically reasonable values of  $\lambda_1, \lambda_2$ .

- (d) Focus on the first case and determine what is the most general form of  $\langle H \rangle$ . Then, the breaking eq. (2) should be obvious. What is the generator of the  $U(1)$ ? Compare this to your result for  $Q$  in exercise H 11.1 (a).