General Warped Solution in 6d Supergravity

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H. M. Lee, CL, JHEP **01**(2006) 062 [arXiv:hep-th/0510026]

Motivation

- Warped $4 + d$-dimensional geometries of the form

$$\mathrm{d}s_d^2 = W^2(y)\mathrm{d}s_4^2 + \mathrm{d}s_d^2$$

  can alleviate the hierarchy problem

- Codimension-two branes are conical, that is, the codimensional mnetric around the brane position is

$$\mathrm{d}s_2^2 \propto \mathrm{d}\rho^2 + \beta^2\rho^2\mathrm{d}\theta^2 \,,$$

  which means there is a deficit angle $2\pi\,(1-\beta)$. These branes do not induce curvarure in the bulk – there is only a δ-function singularity.

- Curvature of codimension-two branes is independent of their tension ("Self-tuning")

Contents

The Model

Ingredients:

- 6d supergravity: Gravity & tensor Multiplet $(G_{MN}, \Psi_M, \chi, B_{MN}, \Phi)$
- Gauged $U(1)_R$-symmetry: vector multiplet (A_M, λ)
- 4d branes with tensions Λ_i

Action: Bulk and branes

$$S_{\text{bulk}} = \int \mathrm{d}^6 X \sqrt{-G}\, \Big\{ \tfrac{1}{2} R - \tfrac{1}{2}\partial_M \Phi \partial^M \Phi - \tfrac{1}{4} e^{-\Phi} F_{MN} F^{MN} - 2g^2 e^{\Phi} + \text{2-form} + \text{fermions} \Big\}$$

$$S_{\text{branes}} = -\sum_i \int \mathrm{d}^4 x_i \sqrt{-g_i}\, \Lambda_i$$

Field and Einstein Equations

Ansatz: Warped background solution with 4d maximal symmetry:

$$\mathrm{d}s^2 = W^2(y)\tilde{g}_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu + \hat{g}_{mn}\mathrm{d}y^m \mathrm{d}y^n\,, \qquad R_{\mu\nu}\left(\tilde{g}\right) = 3\lambda \tilde{g}_{\mu\nu}$$

$$F_{mn} = \sqrt{\hat{g}}\,\epsilon_{mn}F(y)\,, \quad F_{\mu\nu} = F_{\mu m} = 0\,, \quad H_{MNP} = 0$$

$$\Phi = \Phi(y)$$

This leads to field and Einstein Equations:

$$\begin{aligned}
D_m\left(W^4 e^{-\Phi}F\right) &= 0 \\
W^{-4}D_m\left(W^4 D^m\Phi\right) &= -\left(\tfrac{1}{2}F^2 e^{-\Phi} - 2g^2 e^{\Phi}\right) \\
3\lambda - \frac{1}{4}W^{-2}D_m D^m W^4 &= -\left(\tfrac{1}{4}F^2 e^{-\Phi} - g^2 e^{\Phi}\right)W^2 \\
R_{mn}\left(\hat{g}\right) - 4W^{-1}D_m D_n W &= D_m\Phi D_n\Phi + \left(\tfrac{3}{4}F^2 e^{-\Phi} + g^2 e^{\Phi}\right)\hat{g}_{mn} \\
&\quad + \sum_i \frac{\Lambda_i}{\sqrt{\hat{g}}}\hat{g}_{mn}\delta^2\left(y - y_i\right)
\end{aligned}$$

Solution

Results:

- $\lambda = 0$, i.e. $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$: Minkowski space is the unique maximally symmetric background [Gibbons et al.]
- Dilaton: $\Phi = \Phi_0 - 2\ln W$
- Gauge flux $F(y) = f e^{\Phi} W^{-4} = f e^{\Phi_0} W^{-6}$
- For the warp factor, rewrite metric in terms of complex coordinate $z = y_5 + \mathrm{i}y_6$ as $(W \equiv e^B)$

$$\mathrm{d}s^2 = e^{2B(z,\bar{z})}\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2A(z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z}\right)$$

- Einstein equations become $((\mu\nu)$, (zz) and $(z\bar{z})$ components)

$$-4e^{-2A}\left(\partial B\bar{\partial}B + \partial\bar{\partial}B\right) = e^{\Phi_0}\left(g^2 - \tfrac{1}{4}f^2 e^{-8B}\right)$$

$$\bar{\partial}V(z) \equiv \bar{\partial}\left(e^{-2A}\bar{\partial}B\right) = 0$$

$$-6\partial\bar{\partial}B - 8\partial B\bar{\partial}B - 2\partial\bar{\partial}A = \tfrac{1}{2}e^{2A}e^{\Phi_0}\left(g^2 + \frac{3}{4}f^2 e^{-8B}\right) + \sum_i \Lambda_i \delta^2\left(z - z_i\right)$$

Solution

Free holomorphic function $V(z)$ determines warp factor:

- Unwarped solution $\Leftrightarrow B = \text{const.} \Leftrightarrow V = 0$ and $f^2 = 4g^2$

  [Aghababaie et al., Redi]

- $V \neq 0 \ \Rightarrow$ ordinary differential equation for the warp factor in terms of new coordinate: [Chodos, Poppitz]

$$\zeta = \tfrac{1}{2} \int^z \frac{\mathrm{d}\omega}{V(\omega)} + \text{c.c} \qquad \frac{\mathrm{d}W}{\mathrm{d}\zeta} = -\gamma^2 \frac{W^4 - 2v + u^2 W^{-4}}{2W^3} = \frac{P(W)}{W^3}$$

with parameters

$$\gamma^2 = \tfrac{1}{4} e^{\Phi_0} g^2 \,, \ u^2 = \frac{f^2}{4g^2} \,, \ v$$

Solution

This can be integrated to give

$$\frac{\left(W^4(\zeta) - W_-^4\right)^{W_-^4}}{\left(W_+^4 - W^4(\zeta)\right)^{W_+^4}} = \exp\left\{2\gamma^2\left(W_+^4 - W_-^4\right)(\zeta - \zeta_0)\right\}$$

with the roots of $P(W)$

$$W_\pm^4 = v \pm \sqrt{v^2 - u^2}$$

- Warp factor bounded in the range $W_- \leq W \leq W_+$
- Reality of warp factor gives constraints $v^2 \geq u^2\,, \quad v > 0$
- Extrema $W_\pm$ reached for $\zeta \to \pm\infty$, correspond to conical singularities
- Four real integration constants: f, Φ_0, v and ζ_0

General Metric

The general metric finally is

$$\mathrm{d}s^2 = W^2\left(z,\bar{z}\right)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{1}{2\left|V(z)\right|}\frac{P(W)}{W^2}\mathrm{d}z\mathrm{d}\bar{z}$$

- Constraint on $V(z)$: For well-behaved (i.e. single-valued and free from singularities up to δ-functions) warp factor and higher curvature invariants, V can have only simple zeroes or poles
- Zeroes and poles of $V(z)$:
  - Simple zero $V(z) \propto \left(z - z_0\right)/c$ with real c leads to conical brane with deficit angle

    $$2\pi\left(1 - \gamma^2\left|c\right|W_\pm^4\left(W_+^4 - W_-^4\right)\right) \quad \text{for } c \lessgtr 0$$

  - Simple pole $V(z) \propto 1/\left(z - z_0\right)$ leads to brane with fixed deficit angle -2π
- Behaviour at infinity: For $V \propto z^n$ with $n > 2$, there will another brane with fixed deficit angle $2\pi\left(2 - n\right)$ at $z = \infty$

Example: Two Branes

Simple ansatz:

$$V(z) = -\frac{z}{c}\,, \quad c \text{ real and positive}$$

- Globally well-defined change of coordinates

$$\zeta = -\tfrac{1}{2} c \ln |z|^2 \qquad\qquad \theta = -\frac{1}{2\mathrm{i}} c \ln \frac{z}{\bar{z}}$$

- Conical branes at $z = 0$ and $z = \infty$ with warp factors $W_\pm \Rightarrow$ warped rugby
- Warp factor ($\mathrm{d}\eta = c^{-1} W^{-4} \mathrm{d}\zeta$)

$$W^4(\eta) = \tfrac{1}{2}\left(W_+^4 + W_-^4\right) + \tfrac{1}{2}\left(W_+^4 - W_-^4\right) \tanh\left[\left(W_+^4 - W_-^4\right) \gamma^2 c \eta\right]$$

  interpolates between W_+ and W_-
- Warp factor does not depend on $\theta \rightsquigarrow$ axial symmetry in extra dimensions

Planck Mass and Flux Quantisation

Four-dimensional Planck mass

$$M_{\mathrm{P}} = \frac{\pi}{2} v c \sqrt{1 - \left(\frac{f}{2gv}\right)^2} M_6^4 = \frac{\pi}{4} c \left(W_+^4 - W_-^4\right) M_6^4$$

For compact extra dimensions, flux is quantised, in this case

$$\frac{W_+^4 - W_-^4}{W_+^4 W_-^4} f = \frac{8n}{g} \frac{e^{-\Phi_0}}{c}$$

This leads to a fine-tuning condition between brane tensions:

$$(2\pi - \Lambda_+)(2\pi - \Lambda_-) = (2\pi n)^2$$

The parameter c can be absorbed by a rescaling of θ, two parameters are fixed by matching of brane tensions to deficit angles $\rightsquigarrow$ One undetermined modulus remains.

Unwarped Limit

- In the unwarped limit, $V \to 0$, so ζ coordinate is not well-defined, but limit is possible:
- For this limit, take c to infinity while keeping

$$k = c \left(W_+^4 - W_-^4 \right)$$

finite $\rightsquigarrow$ unwarped rugby, two branes with same deficit angle

$$\beta = \gamma^2 W_+^{-4} k$$

and equal tensions.

- This is consistent with brane conditions, and Planck mass stays finite.

Example: Many Branes

For a multi-brane solution, take a similar ansatz:

$$V = \frac{1}{c} \prod_{i=1}^{N} (z - z_i)$$

- For single-valued warp factor, c and all z_i have to be real
- N branes with warp factor W_+ or W_-, depending on the sign of

$$a_i = c \prod_{j \neq i} \frac{1}{z_i - z_j}$$

- Additional brane at $z = \infty$ with fixed brane tension $\Lambda^\infty = 2\pi (2 - N)$
- After flux quantisation and brane tension matching, still one undetermined modulus

Conclusions and Outlook

- We have presented the general warped solution of 6d supergravity with 4d maximal symmetry
- Important properties depend on a free holomorphic function
  - Linear function: Recover known two-brane solutions
  - Function with many zeros gives multi-brane solutions. Howver, fixed-tension brane required in this case
- One undetermined modulus in simple cases, one fine-tuning relation between brane tensions
- To do:
  - Systematic study of different functions
  - In particular: Elliptic (doubly periodic) functions for torus geometry in extra dimensions