Fixing D7 Brane Positions by F-Theory Fluxes

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Motivation

- F-Theory: Nonperturbative version of type IIB string theory
  \[\text{[Vafa;Sen]}\]

- Add two auxiliary dimensions, singularities of compactification manifold encode brane positions

- Recently, lots of interest in F-theory for model building interest
  \[\text{[Beasley, Heckman, Vafa; Saulina, Schäfer-Nameki; Bourjaily; Tatar, Watari...]}\]

- Local models do not address global constraints like tadpole cancellation
  - Four-form flux can stabilise moduli, including brane positions
  - Simple example: F-Theory on \(K3 \times \widetilde{K3}\), where is an elliptic fibration over \(\mathbb{P}^1\)
    \[\text{[Görlich et al.; Lust et al.; Aspinwall, Kallosh; Dasgupta et al.]}\]
  - Includes as special case the type IIB orientifold \(K3 \times T^2/\mathbb{Z}_2\)
    \[\text{[Angelantonj et al.]}\]
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F-Theory/M-Theory Duality

Fibrewise duality: $X_6 \times T^2 \leadsto$ elliptically fibred CY$_4$
dual to type IIB on base of fibration
F-Theory/M-Theory Duality

\[ \text{M-Theory on } X_6 \times T^2 \xrightarrow{\text{Compactify on one } S^1} \text{Type IIA on } X_6 \times S^1_A \xrightarrow{T\text{-dualise along } S^1_A} \text{Type IIB on } X_6 \times S^1_B (R_B = 1/R_A) \]

\[ R_B \rightarrow \infty \]
\[ (R_A \rightarrow 0) \]

Type IIB on \( X_6 \)
Four noncompact dimensions

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M-Theory on $X_6 \times T^2$ \rightarrow\hspace{1cm} \text{Compactify one $S^1$} \rightarrow\hspace{1cm} \text{Type IIA on $X_6 \times S^1_A$} \rightarrow\hspace{1cm} \text{T-dualise along $S^1_A$} \rightarrow\hspace{1cm} \text{Type IIB on $X_6 \times S^1_B$ (}$R_B = 1/R_A$)$

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M-Theory on $X_6 \times T^2$ \rightarrow\hspace{1cm} \text{Shrink one $S^1$} \rightarrow\hspace{1cm} \text{Vol} (T^2) \rightarrow 0 \rightarrow\hspace{1cm} \text{M-Theory/F-Theory Duality} \rightarrow\hspace{1cm} \text{Type IIB on $X_6$} \rightarrow\hspace{1cm} \text{Four noncompact dimensions}$

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M-Theory on \( X_6 \times T^2 \) Compactify one \( S^1 \) \( \rightarrow \) Type IIA on \( X_6 \times S^1_A \) \( T \)-dualise along \( S^1_A \) \( \rightarrow \) Type IIB on \( X_6 \times S^1_B \) \( (R_B = 1/R_A) \)

Shrink one \( S^1 \)

M-Theory on \( X_6 \times T^2 \) \( \rightarrow \) M-Theory/F-Theory Duality \( \rightarrow \) Type IIB on \( X_6 \) Four noncompact dimensions

\( \text{Vol}(T^2) \rightarrow 0 \)

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K3: Calabi–Yau Two-Fold

- $H^2(K3, \mathbb{R})$ has signature $(3, 19)$
- Holomorphic two-form and Kähler form spanned by three real forms $\omega_i$ with $\omega_i \cdot \omega_j = \delta_{ij}$ and overall volume $\nu$:

$$\omega = \omega_1 + i\omega_2$$

$$j = \sqrt{2\nu} \omega_3$$

- K3 is hyperkähler, i.e. $SO(3)$ rotating the $\omega_i \mapsto$ geometry fixed by positive-norm three-plane $\Sigma \subset H^2(K3, \mathbb{R})$ and $\nu$
- Moduli space has $3 \times 19 + 1 = 58$ dimensions
- Integral basis for $H^2(K3)$ with intersection matrix

$$U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8), \text{ where } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } E_8 \text{ is Cartan matrix of } E_8$$

$\Rightarrow$ The $\omega_i$ must have components along the $U$ blocks, components along “$E_8$ directions” determine gauge group.
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K3: Elliptic Fibration and F-Theory Limit

- For an elliptically fibred $K3$, require integral cycles $B$ and $F$ (base and fibre) with
  - intersection matrix $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$
  - $B \cdot \omega = F \cdot \omega = 0$

$\Rightarrow$ $(B, F)$ spans a $U$ block, and we can parametrise the Kähler form as

$$j = b B + f F + c^a u_a \quad (\text{where } u_a \cdot \omega = 0)$$

- F-theory limit: Fibre volume shrinks to zero $\Rightarrow b \to 0$. $K3$ volume is $\nu \sim b f - c^a c^a$, so we have to take $c^a \to 0$ as fast as $\sqrt{b}$ (as intuitively expected).
- In the limit, $j = f F$ is the Kähler form of the $\mathbb{P}^1$ base.
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Cycles Between Branes

- One leg in the base, one in the fibre torus
- **Shrink to zero** when the branes are moved on top of each other.
- They are topologically a sphere ↔ self-intersection $-2$.
- Cycles meeting at a brane intersect once, cycles encircling O planes ($\times$) do not intersect.
Shrinking Cycles and Gauge Enhancement

Intersection matrix of shrinking cycles determines gauge group:
Consider e.g. $T^2/\mathbb{Z}_2$ orientifold: One O7, four D7s $\leadsto SO(8)$

\[
\begin{pmatrix}
-2 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 \\
0 & 1 & 0 & -2 \\
1 & 0 & 1 & 1
\end{pmatrix}
\]

In appropriate basis, complex structure of $\tilde{K}3$ is

\[
\omega = \frac{\alpha}{2} + u e_2 + s \frac{\beta}{2} - \left( u s - \frac{z^2}{2} \right) e_1 + z I \hat{E}_I
\]

Explicit mapping between complex structure and brane positions!

[Braun, Hebecker, Triendl]
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base complex structure, axiodilaton, brane positions, $z_I = 0$ is $SO(8)^4$ point

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Flux Potential

- **Type IIB:** Three-form flux $G_3$ on the bulk, two-form gauge flux $F_2$ on the branes can stabilise geometric and brane moduli.
- In M-theory, these are combined into *four-form flux* $G_4$ (brane moduli become four-form geometric moduli).

**Consistency conditions:**
- Flux quantisation: flux needs to be integral.
- Tadpole cancellation (without spacetime-filling M2 branes):
  \[
  \frac{1}{2} \int_{K3 \times \tilde{K3}} G_4 \wedge G_4 = \frac{\chi}{24} = 24
  \]
- $G_4$ needs to have exactly one leg on each on base and fibre for Lorentz invariance, hence two on each $K3$: $G = G^{I \wedge \eta_I \wedge \tilde{\eta}_\Lambda}$, but no flux along $B$ or $F$. 
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Potential

- Flux potential ($V$ is the volume):

\[ V = \frac{1}{4V^3} \left( \int_{K3 \times \tilde{K}3} G \wedge *G - \frac{\chi}{12} \right) \]

- $K3 \times \tilde{K}3$ is not a proper CY$_4$: Holonomy is $SU(2) \times SU(2)$

- $G_4$ induces map $G : H^2(\tilde{K}3) \rightarrow H^2(K3)$ and its adjoint $G^a$ by

\[
G\tilde{\eta} = \int_{\tilde{K}3} G \wedge \tilde{\eta} \quad \quad \quad G^a \eta = \int_{K3} G \wedge \eta
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- Potential is concisely expressed in terms of these maps

[Haack,Louis]
Potential

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- Potential is concisely expressed in terms of these maps
$K3 \times \tilde{K}3$ Flux Potential

\[
V = -\frac{1}{2(\nu \cdot \tilde{\nu})^3} \left( \sum_j \| G \tilde{\omega}_j \|^2_\perp + \sum_i \| G^a \omega_i \|^2_\perp \right)
\]

Here $\| \cdot \|^2_\perp$ is the norm orthogonal to the three-plane

- Positive definite potential
- Manifestly symmetric under $SO(3)$
- Minima at $V = 0$:

$G \tilde{\omega}_j \in \langle \omega_1, \omega_2, \omega_3 \rangle \quad G^a \omega_i \in \langle \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3 \rangle$

- $\nu$ and $\tilde{\nu}$ are unfixed, flat directions (when $V = 0$)
K3 × ˜K3 Flux Potential

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- \( \nu \) and \( \tilde{\nu} \) are unfixed, flat directions (when \( V = 0 \))
• Minkowski minima do not necessarily exist: $G^a G$ must be diagonalisable and positive semi-definite (not guaranteed although $G^a G$ is self-adjoint, since metric is indefinite!)

• Flat directions generally exist and are desired: M-theory moduli become part of 4D vector fields in F-theory limit $\rightsquigarrow$ fixing these moduli breaks the gauge group (rank-reducing)

• Flux also induces explicit mass term for three-dimensional vectors

• Vacua can preserve $\mathcal{N} = 4$, $\mathcal{N} = 2$ or $\mathcal{N} = 0$ supersymmetry in four dimensions, depending on the action of $G$ on the three-plane
Stabilisation Strategy

- F-theory limit fixes Kähler form (up to base volume), $j = f F$
- Holomorphic two-form determines shrinking cycles, i.e. gauge enhancement

To stabilise a desired brane configuration:
- Identify set of shrinking cycles to obtain desired brane stacks
- Chose these as part of a basis of $H^2(K3)$ and complete by integral cycles
- Find an integral block-diagonal flux that satisfies tadpole cancellation condition (strong constraint and computationally costly)
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Examples

We give explicit examples of

- The $T^2/\mathbb{Z}_2$ orientifold with $SO(8)^4$: Four stacks of four D7 branes and one O7 plane each

  - Moving one brane off a stack.
    $\Rightarrow SO(8)^3 \times SO(6) \times U(1)$ or
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Conclusion

- We have a nice geometric picture of D7 brane motion
- We found the flux potential in M-theory and explicit conditions for the existence of minima and gauge symmetry breaking
- Translation to F-theory $\Rightarrow$ recipe to find fluxes that stabilise a desired situation
- Explicit examples: We can move any brane

- Open problem: Numerical scan of matrices is very time–consuming
- Outlook: Generalise to elliptically fibred four-folds to get physically more realistic models, in particular intersecting branes
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