Physikalisches Institut
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Theoretische Physik

Exercises 0
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Exercises on Theoretical Particle Physics II

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Class Exercise

0.1 Weyl spinors and Grassmann variables

In this exercise, we want to illustrate the connection between spinors and Grassmann variables and get used to the spinor index conventions.
Let θ_α, $\alpha = 1, 2$ be anti-commuting complex Grassmann variables

$$\{\theta_\alpha, \theta_\beta\} = 0. \tag{1}$$

As left-chiral Weyl spinors, they transform in the $(1/2, 0)$ representation of the Lorentz group

$$\theta \mapsto (D_L)\theta = \exp\left[-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right]\theta \tag{2a}$$

$$\bar{\theta} \mapsto (D_R)\bar{\theta} = \exp\left[-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right]\bar{\theta} \tag{2b}$$

with the Pauli matrices $\sigma^\mu := (\mathbb{1}, \sigma^i)$, $\bar{\sigma}^\mu := (\mathbb{1}, -\sigma^i)$ and the spin generators $\sigma^{\mu\nu} := \frac{\mathrm{i}}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$, $\bar{\sigma}^{\mu\nu} := \frac{\mathrm{i}}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$.

(a) Let $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$ be the totally antisymmetric 2×2 tensor, i.e. $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ with normalization $\epsilon_{12} = 1$. Show that: $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = -\delta_\alpha^\gamma$.

(b) On last year's exercise you proved

$$\sigma_2 = (D_L)^T\sigma_2 D_L. \tag{3}$$

Why does this means that σ_2 is a spinor metric? Being a metric, it can be used to raise and lower spinor indices.

(c) Show that $\epsilon = \mathrm{i}\sigma_2$ is an equivalent choice for the metric. From now on, we use ϵ to raise and lower spinor indices:

$$\theta^\alpha := -\epsilon^{\alpha\beta}\theta_\beta. \tag{4}$$

Give the inverse of this relation.

(d) We define the conjugate Grassmann variable $\bar{\theta}^{\dot{\alpha}}$ as $\bar{\theta}^{\dot{\alpha}} := (\theta^\alpha)^*$. Verify (2b), i.e. show that it transforms in the $(0, 1/2)$ representation of the Lorentz group (as a right-chiral Weyl spinor).
Hint: You showed last year that $\sigma_2 D_L \sigma_2 = D_R^*$. Use this to calculate the transformation of $\epsilon^{\alpha\beta}\theta_\beta$.

(e) The conventions for contracting spinor indices are:

$$\xi\psi := \xi^\alpha\chi_\alpha \text{ and } \bar{\xi}\bar{\chi} := \bar{\xi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \tag{5}$$

Verify the following identities:

(i)	$\xi^\alpha\chi_\alpha = -\xi_\alpha\chi^\alpha$	and	$\bar{\xi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\xi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}$
(ii)	$\xi\chi = \chi\xi$	and	$\bar{\xi}\bar{\chi} = \bar{\chi}\bar{\xi}$

(f) Prove furthermore:

$$\begin{array}{lrcl l rcl} \text{(i)} & \theta^\alpha\theta^\beta & = & \frac{1}{2}\epsilon^{\alpha\beta}\theta\theta & \text{and} & \theta_\alpha\theta_\beta & = & \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \\ \text{(ii)} & \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} & = & -\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta & \text{and} & \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} & = & -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta \end{array}$$

In summary, the components of spinors are Grassmann variables. They anti-commute and transform in the correct Lorentz representations.

0.2 Weyl spinors and Pauli matrices

This exercise is intended to further establish the relation between the Pauli matrices, the spinor metric, and spinors as Grassmann variables.

(a) Use the Lorentz transformations (2) to deduce the spinor index structure for the Pauli matrices σ^μ, $\bar\sigma^\mu$ and the spin generators $\sigma^{\mu\nu}$, $\bar\sigma^{\mu\nu}$ in terms of the conventions introduced in (5).

(b) Check the following identites:

$$\begin{array}{lrcl} \text{(i)} & (\bar\sigma^\mu)^T & = & -\epsilon\sigma^\mu\epsilon \\ \text{(ii)} & (\sigma^\mu)^{\alpha\dot\beta} & = & (\bar\sigma^\mu)^{\dot\beta\alpha} \end{array}$$

(c) Verify furthermore:

$$\begin{array}{lrcl} \text{(i)} & \bar\xi\bar\sigma^\mu\chi & = & -\chi\sigma^\mu\bar\xi \\ \text{(ii)} & \chi_\alpha\bar\xi_{\dot\beta} & = & \frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}(\chi\sigma_\mu\bar\xi) \\ \text{(iii)} & (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) & = & \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta) \end{array}$$

Hint: $\eta_{\mu\nu}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$ *(Prove it)*

0.3 Grassmann variable calculus

This exercise is intended to introduce differentiation and integration of Grassmann variables and to investigate the consequences.

The Grassmann differentiation is defined as

$$\partial_\alpha := \frac{\partial}{\partial\theta^\alpha} \text{ and } \bar\partial^{\dot\alpha} := \frac{\partial}{\partial\bar\theta_{\dot\alpha}}, \tag{6}$$

with the usual relation $\partial_\alpha\theta^\beta = \delta^\beta_\alpha$ and $\bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}_{\dot\beta}$. However, the product rule must be defined with a minus sign:

$$\partial_\alpha(\theta^\beta\theta^\gamma) = \delta^\beta_\alpha\theta^\gamma - \theta^\beta\delta^\gamma_\alpha.$$

(a) Show that:

$$\begin{array}{lrcl} \text{(i)} & \partial^\alpha & = & \epsilon^{\alpha\beta}\partial_\beta \\ \text{(ii)} & \partial^\alpha\partial_\alpha(\theta\theta) & = & \bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}(\bar\theta\bar\theta) = 4 \end{array}$$

The Grassman integration is defined as

$$\int d\theta^\alpha := 0 \text{ and } \int d\theta^\alpha \theta_\beta := \delta^\alpha_\beta, \tag{7}$$

which is linear. Note in particular that integration and differentiation is the same operation for Grassmann variables. The volume elements are

$$d^2\theta := -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta}, \quad d^2\bar{\theta} := -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}, \quad d^4\theta := d^2\theta d^2\bar{\theta}. \tag{8}$$

From this, we find

$$\int d^2\theta (\theta\theta) = \int d^2\bar{\theta}(\bar{\theta}\bar{\theta}) = 1. \tag{9}$$

(b) Owing to the nilpotence of Grassmann variables stemming from (1), the Taylor series expansion of any function $f(\theta, \bar{\theta})$ is finite. Write down the Taylor expansions of $f(\theta)$ and $f(\theta, \bar{\theta})$ and determine $\int d^2\theta f(\theta)$ and $\int d^4\theta f(\theta, \bar{\theta})$ in terms of their Taylor series expansion coefficients. Expressions like these appear in the SUSY action when formulated in superspace.

Integrations over Grassmann variables also play an important role in QFT. They are used e.g. in the Feynman path integral for fermions. Besides, they are important in the gauge fixing procedure for non-Abelian gauge fields. There, Faddeev–Popov ghosts fields $c, \bar{c}$ (scalar fields which *anti*commute) are introduced to fix the gauge. The integrals that appear in QFT in the partition function after Wick-rotation and completing the square are of the form

$$I := \int \mathrm{d}^N x \exp[-x_j A_{jk} x_k], \tag{10}$$

with an $N \times N$ matrix A (the propagator, Faddeev-Popov matrix, ...).

(c) Assume A is symmetric. Prove $I = \left(\frac{\pi^N}{\det(A)}\right)^{\frac{1}{2}}$ where the integral runs from $-\infty$ to ∞.

(d) Now we replace the 'bosonic' coordinate x_i by complex Grassmann variables θ_i and let A be a generic matrix. Compute

$$\int \mathrm{d}^N\theta \mathrm{d}^N\bar{\theta} \; \exp[-\bar{\theta}_j A_{jk} \theta_k] \,.$$