Physikalisches Institut
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Theoretische Physik

Exercises 1
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Exercises on Theoretical Particle Physics II

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1.1 Weyl spinors and Grassmann variables (*9 credits*)

In this exercise, we want to illustrate the connection between spinors and Grassmann variables and get used to the spinor index conventions.
Let θ_α, $\alpha = 1, 2$ be anti-commuting Grassmann variables

$$\{\theta_\alpha, \theta_\beta\} = 0. \tag{1}$$

As left-chiral Weyl spinors, they transform in the $(1/2, 0)$ representation of the Lorentz group

$$\theta \mapsto (D_L)\theta = \exp\left[-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right]\theta \tag{2a}$$

$$\bar\theta \mapsto (D_R)\bar\theta = \exp\left[-\frac{i}{2}\omega_{\mu\nu}\bar\sigma^{\mu\nu}\right]\bar\theta \tag{2b}$$

with the Pauli matrices $\sigma^\mu := (\mathbb{1}, \sigma^i)$, $\bar\sigma^\mu := (\mathbb{1}, -\sigma^i)$ and the spin generators $\sigma^{\mu\nu} := \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$, $\bar\sigma^{\mu\nu} := \frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)$.

(a) Let $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$ be the totally antisymmetric 2×2 tensor, i.e. $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ with normalization $\epsilon_{12} = 1$. Show that: $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = -\delta_\alpha^\gamma$. (*1 credit*)

(b) On last year's exercise sheet 2, you proved in equation (19)

$$\sigma_2 = (D_L)^T\sigma_2 D_L. \tag{3}$$

Why does this means that σ_2 is a spinor metric. Being a metric, it can be used to raise and lower spinor indices. (*1 credit*)

(c) Show that $\epsilon = i\sigma_2$ is an equivalent choice for the metric. From now on, we use ϵ to raise and lower spinor indices:

$$\theta^\alpha := -\epsilon^{\alpha\beta}\theta_\beta. \tag{4}$$

Give the inverse of this relation. (*1 credit*)

(d) We define the conjugate Grassmann variable $\bar\theta^{\dot\alpha}$ as $\bar\theta^{\dot\alpha} := (\theta^\alpha)^*$. Verify (2b), i.e. show that it transforms in the $(0, 1/2)$ representation of the Lorentz group (as a right-chiral Weyl spinor).
Hint: You showed last year that $\sigma_2 D_L \sigma_2 = D_R^*$. Use this to calculate the transformation of $\epsilon^{\alpha\beta}\theta_\beta$. (*2 credits*)

(e) The conventions for contracting spinor indices are:

$$\xi\psi := \xi^\alpha\chi_\alpha \text{ and } \bar\xi\bar\chi := \bar\xi_{\dot\alpha}\bar\chi^{\dot\alpha} \tag{5}$$

Verify the following identities: (*2 credits*)

$$\begin{array}{lrclcrcl} \text{(i)} & \xi^\alpha\chi_\alpha & = & -\xi_\alpha\chi^\alpha & \text{and} & \bar\xi_{\dot\alpha}\bar\chi^{\dot\alpha} & = & -\bar\xi^{\dot\alpha}\bar\chi_{\dot\alpha} \\ \text{(ii)} & \xi\chi & = & \chi\xi & \text{and} & \bar\xi\bar\chi & = & \bar\chi\bar\xi \end{array}$$

(f) Prove furthermore: (*2 credits*)

$$\begin{array}{lrcllrcl} \text{(i)} & \theta^\alpha\theta^\beta & = & \frac{1}{2}\epsilon^{\alpha\beta}\theta\theta & \text{and} & \theta_\alpha\theta_\beta & = & \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \\ \text{(ii)} & \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} & = & -\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta & \text{and} & \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} & = & -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta \end{array}$$

In summary, the components of spinors are Grassmann variables. They anti-commute and transform in the correct Lorentz representations.

1.2 Weyl spinors and Pauli matrices (*6 credits*)

This exercise is intended to further establish the relation between the Pauli matrices, the spinor metric, and spinors as Grassmann variables.

(a) Use the Lorentz-transformations (2) to deduce the spinor index structure for the Pauli matrices σ^μ, $\bar\sigma^\mu$ and the spin generators $\sigma^{\mu\nu}$, $\bar\sigma^{\mu\nu}$ in terms of the conventions introduced in (5). (*1 credit*)

(b) Check the following identites: (*2 credits*)

$$\begin{array}{lrcl} \text{(i)} & (\bar\sigma^\mu)^T & = & (-i\sigma_2)\sigma^\mu(i\sigma_2) \\ \text{(ii)} & (\sigma^\mu)^{\alpha\dot\beta} & = & (\bar\sigma^\mu)^{\dot\beta\alpha} \end{array}$$

(c) Verify furthermore: (*3 credits*)

$$\begin{array}{lrcl} \text{(i)} & \bar\xi\bar\sigma^\mu\chi & = & -\chi\sigma^\mu\bar\xi \\ \text{(ii)} & \chi_\alpha\bar\xi_{\dot\beta} & = & \frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}(\chi\sigma_\mu\bar\xi) \\ \text{(iii)} & (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) & = & \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta) \end{array}$$

1.3 Grassmann variable calculus (*8 credits*)

This exercise is intended to introduce differentiation and integration of Grassmann variables and to investigate the consequences.

The Grassmann differentiation is defined as

$$\partial_\alpha := \frac{\partial}{\partial\theta^\alpha} \text{ and } \bar\partial^{\dot\alpha} := \frac{\partial}{\partial\bar\theta_{\dot\alpha}}, \tag{6}$$

with the usual relation $\partial_\alpha\theta^\beta = \delta_\alpha^\beta$ and $\bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}_{\dot\beta}$. However, the product rule must be defined with a minus sign:

$$\partial_\alpha(\theta^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma - \theta^\beta\delta_\alpha^\gamma.$$

(a) Show that: (*2 credits*)

$$\begin{array}{lrcl} \text{(i)} & \partial^\alpha & = & \epsilon^{\alpha\beta}\partial_\beta \\ \text{(ii)} & \partial^\alpha\partial_\alpha(\theta\theta) & = & \bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}(\bar\theta\bar\theta) = 4 \end{array}$$

The Grassman integration is defined as

$$\int d\theta^\alpha := 0 \text{ and } \int d\theta^\alpha \theta_\beta := \delta^\alpha_\beta, \tag{7}$$

which is linear. Note in particular that integration and differentiation is the same operation for Grassmann variables. The volume elements are

$$d^2\theta := -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta}, \quad d^2\bar{\theta} := -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}, \quad d^4\theta := d^2\theta d^2\bar{\theta}. \tag{8}$$

From this, we find

$$\int d^2\theta(\theta\theta) = \int d^2\bar{\theta}(\bar{\theta}\bar{\theta}) = 1. \tag{9}$$

(b) Owing to the nilpotence of Grassmann variables stemming from (1), the Taylor series expansion of any function $f(\theta, \bar{\theta})$ is finite. Determine $\int d^2\theta f(\theta)$ and $\int d^4\theta f(\theta, \bar{\theta})$ in terms of their Taylor series expansion coefficients. Expressions like this appear in the SUSY action when formulated in superspace. (*1 credit*)

Integration over Grassmann variables also play an important role in QFT. They are used e.g. in the Feynman path integral for fermions. Besides, they are important in the gauge fixing procedure for non-Abelian gauge fields. There, Faddeev-Popov ghosts fields $c, \bar{c}$ (bosonic fields which *anti*commute) are introduced to fix the gauge. The integrals that appear in QFT in the partition function after Wick-rotation and completing the square are of the form

$$I := \prod_{i=1}^{N} \int dx_i \exp[-x_j A_{jk} x_k], \tag{10}$$

with an $N \times N$ matrix A (the propagator, Faddeev-Popov matrix, ...).

(c) Assume A is symmetric. Prove $I = \left(\frac{\pi^N}{\det(A)}\right)^{\frac{1}{2}}$ where the integral runs form $-\infty$ to ∞.

Hint: Use the spectral theorem to choose a convenient basis for A. Then use the Gamma function $\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}$ to perform the integration. Remember $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (*3 credits*)

(d) Replace now the 'bosonic' x_i in (10) by Grassmann variables θ_i and let A be an antisymmetric matrix. What is I now?
Hint: Expand the exponential in a Taylor series. Do the calculations first for small N. (*2 credits*)

This is an example for possible cancellations that can appear between bosonic and fermionic fields. This is used in the Faddeev-Popov procedure to cancel zero mode contributions which are linked to the fact that the gauge is not fixed. In SUSY, contributions from fields that transform with opposite statistics prevent e.g. radiative corrections to the Higgs mass.