Physikalisches Institut
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Theoretische Physik

Exercise 9
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WS 10/11

Exercises on Theoretical Particle Physics

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–Home Exercises–
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H 9.1 Group-theoretical GUT breaking *6 points*

We believe that the SM gauge group unifies to one simple Lie algebra (e. g. $\mathfrak{su}(5)$) which is broken at very high energies $\mathcal{O}(10^{16}\,\text{GeV})$. Representations of such Grand Unified Theory (GUT) group decompose into those of the SM gauge group. Hence, tools for this group-theoretical symmetry breaking have to be applied.

Dynkin's Symmetry Breaking: To each simple root one assigns an integer number, called the **Kač-label** a_i. They are given as the coefficients of the decomposition of the highest root in the basis of simple roots. Deleting any node with Kac-label $a_i = 1$ from the Dynkin diagram gives a maximal regular subalgebra times a U(1) factor.

(a) In the case of SU(5), all Kač-labels are 1. Apply Dynkin's rule to find the symmetry breaking yielding the SM gauge group, i. e.

$$\text{SU}(5) \to \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)\,.$$

The U(1) generator is constructed as a Cartan element of SU(5) such that it is annihilated by all roots of SU(3) × SU(2). Show that $Q = \text{diag}(-2, -2, -2, 3, 3)$ fulfills these conditions. (*1 point*)

(b) The **5** is a reducible representation of the subgroup SU(3) × SU(2) × U(1). Let α_1 and α_2 correspond to SU(3) and α_4 to SU(2). Thus, every weight λ of SU(5) decomposes as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (\lambda_1, \lambda_2|\lambda_4) = (\mu|\nu).$$

First, write down all weights $(\mu|\nu)$, then find the highest weight μ and determine all weights and the dimension of the corresponding representation. Consider now the values of ν belonging to this μ-representation and state the dimension of the ν-representation! Repeat these steps starting with the highest weight ν. Finally, determine the U(1) charge by applying the U(1) generator to the weight vectors. The result reads

$$\mathbf{5} \to (\mathbf{3}, \mathbf{1})_{-2} \oplus (\mathbf{1}, \mathbf{2})_{3}.$$

(*1 point*)

(c) Repeat the analysis for the representation **10** and verify

$$\mathbf{10} \to (\mathbf{1},\mathbf{1})_6 \oplus (\overline{\mathbf{3}},\mathbf{1})_{-4} \oplus (\mathbf{3},\mathbf{2})_1 .$$

Hint: All weights which appear in the calculation have multiplicity 1. (*1.5 points*)

(d) Perform the breaking for the representation corresponding to the highest weight with Dynkin coefficients $(1,0,0,1)$, i. e. the adjoint **24**. The result reads

$$\mathbf{24} \to (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\overline{\mathbf{3}},\mathbf{2})_5 \oplus (\mathbf{3},\mathbf{2})_{-5} .$$

Identify the gauge bosons of the standard model. (*1.5 points*)

Hint: All weights which appear in the calculation have multiplicity 1, except for $(0,0,0,0)$ *in* **24** *of* $SU(5)$ *with multiplicity 4 and* $(0,0)$ *in* **8** *of* $SU(3)$ *with multiplicity 2. What is the origin of this?*

After a renormalization of the U(1) generator to $Q' = \frac{1}{6}Q$ we recover one family of the standard model in $\overline{\mathbf{5}} \oplus \mathbf{10}$.

H 9.2 Dynamical GUT breaking *4 points*

It is necessary to generalize the Higgs mechanism of the SM to understand the symmetry breaking of any GUT theory to the SM. Thus, we describe the Higgs mechanism for a field H in an arbitrary representation ρ of a semi-simple Lie algebra $\mathfrak{g}$.

(a) Consider a complex scalar H in the representation ρ of a gauge group $\mathcal{G}$. Assume further that H acquires a vev $\langle H \rangle$ due to some potential. Deduce from the kinetic term

$$\left.(D_\mu H)^* (D^\mu H)\right|_{\mathbf{1}} = \left.\left(\partial_\mu H + \mathrm{i} g \rho(T^a) A^a_\mu H\right)^* \left(\partial^\mu H + \mathrm{i} g \rho(T^b) A^{b\mu} H\right)\right|_{\mathbf{1}} ,$$

that a gauge boson A^a_μ is massless, if $\rho(T^a)\langle H \rangle = 0$. Then, T^a belongs to the unbroken gauge group $\mathcal{G}'$.
Specialize to H in the adjoint representation with the kinetic term $\mathrm{Tr}(D_\mu H)^\dagger (D^\mu H)$ and deduce

$$T^a \in \mathcal{G}' \quad \text{if} \quad [T^a, \langle H \rangle] = 0, \qquad T^a \notin \mathcal{G}' \quad \text{if} \quad [T^a, \langle H \rangle] \neq 0 .$$

Let us apply this for the desired symmetry breaking by introducing a Higgs field in the adjoint of SU(5), i. e. a 5×5 hermitian traceless matrix.[1] We work with a scalar potential invariant under $H \to -H$ of the form

$$V(H) = -m^2 \mathrm{Tr}\left(H^2\right) + \lambda_1 \left(\mathrm{Tr}\left(H^2\right)\right)^2 + \lambda_2 \mathrm{Tr}\left(H^4\right) .$$

(*1 point*)

[1] Note that this is not the SM-Higgs field, which is contained in the **5**.

(b) First, use the previous results to argue that a Higgs field H precisely in the adjoint **24** is an appropriate choice to break SU(5) to the SM. Which component of **24** should develop the VEV? (cf. exercise H 9.1 (d)) Use the gauge symmetry $H \to H' = UHU^\dagger$ to obtain

$$H = \mathrm{diag}(h_1, h_2, h_3, h_4, h_5)$$

and check that the minimum of the potential is given by the same equation $\forall h_i$:

$$4\lambda_2 h_i^3 + 4\lambda_1 a h_i - 2m^2 h_i - \mu = 0 \qquad \text{with} \quad a = \sum_j h_j^2, \quad \forall i = 1, \ldots, 5. \tag{1}$$

Here μ is a Lagrange multiplier necessary to impose the constraint $\sum_i h_i = 0$.

The cubic equation (1) has at most three roots denoted by ϕ_1, ϕ_2, ϕ_3. Thus, there are at most three different eigenvalues $h_i \in \{\phi_1, \phi_2, \phi_3\}$. Let n_i be the multiplicity of the eigenvalue ϕ_i, $i = 1, 2, 3$, in $\langle H \rangle$:

$$\langle H \rangle := \mathrm{diag}(\phi_{i_1}, \ldots, \phi_{i_5}) \qquad \text{with} \quad n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0.$$

(*1 point*)

(c) Following part (a), what is the most general symmetry breaking of SU(5)? What happens to the rank of the gauge group? Consider also possible U(1) factors. Depending on the relative magnitude of the parameters λ_1 and λ_2, the combinations $(3, 2, 0)$, $(2, 2, 1)$ or $(4, 1, 0)$ for (n_1, n_2, n_3) minimize the potential. Thus,

$$\texttt{case1:}\ \mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)\,, \quad \texttt{case2:}\ \mathrm{SU}(5) \to \mathrm{SU}(4) \times \mathrm{U}(1)\,, \tag{2}$$

which gives restrictions on phenomenologically reasonable values of λ_1, λ_2. (*1 point*)

(d) Focus on the first case and determine what is the most general form of $\langle H \rangle$. Then, the breaking eq. (2) should be obvious. What is the generator of the U(1)? Compare this to your result for Q in exercise H 9.1 (a). (*1 point*)

H.9.3 Dynkin diagram of $\mathfrak{so}$(2n) — *10 points*

The orthogonal groups are given by matrices which satisfy $A^T A = \mathbb{1}$.

(a) Using the correspondence between elements of the group and elements of the Lie algebra, $A = \exp \mathcal{A} \approx \mathbb{1} + \mathcal{A}$, show that the requirement is:

$$\mathcal{A} + \mathcal{A}^T = 0\,.$$

Clearly these matrices have only off-diagonal elements. As a result, it would be hard to find the Cartan subalgebra as we did for $\mathfrak{su}(n)$ by using diagonal matrices. To avoid this problem, we perform a unitary transformation on the matrices A. (*0.5 points*)

(b) Use the ansatz $A = UBU^\dagger$ with U unitary, define $K = U^T U$ to show that

$$B^T K B = K\,.$$

Furthermore, expand B in the usual way $B = \exp \mathcal{B} \approx \mathbb{1} + \mathcal{B}$ to get the condition:

$$\mathcal{B}^T K + K\mathcal{B} = 0\,. \tag{3}$$

(*1 point*)

(c) A convenient choice for U in the case of $\mathfrak{so}(2n)$ is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i\mathbb{1} & -i\mathbb{1} \\ -\mathbb{1} & -\mathbb{1} \end{pmatrix} ,$$

with $\mathbb{1}$ being the $n \times n$ identity matrix. What is the form of K? *(0.5 points)*

(d) We represent $\mathcal{B}$ in terms of $n \times n$ matrices $\mathcal{B}_i$:

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_3 & \mathcal{B}_4 \end{pmatrix} .$$

Show that from Eq.(3) follows:

$$\mathcal{B}_1 = -\mathcal{B}_4^T , \qquad \mathcal{B}_2 = -\mathcal{B}_2^T , \qquad \mathcal{B}_3 = -\mathcal{B}_3^T .$$

A basis of $2n \times 2n$ matrices fulfilling these conditions is given by $(j, k \leq n)$:

$$\begin{aligned} e^1_{jk} &= e_{j,k} - e_{k+n,j+n} , && \\ e^2_{jk} &= e_{j,k+n} - e_{k,j+n} && j < k , \\ e^3_{jk} &= e_{j+n,k} - e_{k+n,j} && j < k . \end{aligned}$$

A basis for the Cartan subalgebra is given by $h_j = e^1_{jj}$. So, a general element of the Cartan subalgebra can be written as: *(1 point)*

$$h = \sum_i \lambda_i h_i .$$

(e) Determine the eigenvalues of the adjoint of h, i. e.

$$\mathrm{ad}(h)\, e^a_{jk} = [h, e^a_{jk}] = \alpha_{e^a_{jk}}(h)\, e^a_{jk} , \qquad a = 1, 2, 3 .$$

We infer that all roots are given by: *(1.5 points)*

$$\begin{aligned} \alpha_{e^1_{jk}}(h) &= (\lambda_j - \lambda_k) && j \neq k , \\ \alpha_{e^2_{jk}}(h) &= (\lambda_j + \lambda_k) && j < k , \\ \alpha_{e^3_{jk}}(h) &= -(\lambda_j + \lambda_k) && j < k . \end{aligned}$$

(f) Convince yourself that the following roots form a basis of all roots and are furthermore positive and simple:

$$\alpha_i(h) = \lambda_i - \lambda_{i+1} , \quad i = 1 \dots n-1 , \qquad \alpha_n(h) = \lambda_{n-1} + \lambda_n .$$

Hint: Exercise H 8.1(d) *(1 point)*

(g) Show that the Killing form of two elements h and $h\prime$ of the Cartan subalgebra can be written in general as

$$\mathcal{K}(h, h') = 4(n-1) \sum_j \lambda_j \lambda'_j .$$

Hint: Exercise H 8.1(f) *(1.5 points)*

(h) Use the theorem of exercise H 8.1 and the result of the last part to obtain from

$$\mathcal{K}(h_{\alpha_i}, h) = \alpha_i(h)$$

the coefficients $\lambda_j^{\alpha_i}$ of h_{α_i}. *(1.5 points)*

(i) Calculate the Cartan matrix and draw the Dynkin diagram of $\mathfrak{so}(2n)$. *(1.5 points)*

Reminder: SU(5) Dynkin Labels

- **5** of SU(5)

$$(1,0,0,0)\,,\;\; (-1,1,0,0)\,,\;\; (0,-1,1,0)\,,\;\; (0,0,-1,1)\,,\;\; (0,0,0,-1)\,.$$

- **10** of SU(5)

$$\begin{aligned}(0,1,0,0)\,,\;\; (1,-1,1,0)\,,\;\; (-1,0,1,0)\,,\;\; (1,0,-1,1)\,,\;\; (-1,1,-1,1)\,,\\ (1,0,0,-1)\,,\;\; (0,-1,0,1)\,,\;\; (-1,1,0,-1)\,,\;\; (0,-1,1,-1)\,,\;\; (0,0,-1,0)\,.\end{aligned}$$

- **24** of SU(5)

$$\begin{aligned}(1,0,0,1)\,,\;\; (-1,1,0,1)\,,\;\; (1,0,1,-1)\,,\;\; (0,-1,1,1)\,,\;\; (-1,1,1,-1)\,,\\ (1,1,-1,0)\,,\;\; (0,0,-1,2)\,,\;\; (0,-1,2,-1)\,,\;\; (-1,2,-1,0)\,,\;\; (2,-1,0,0)\,,\\ (0,0,0,0)\,,\end{aligned}$$

  and their negatives.