Physikalisches Institut
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Theoretische Physik

Exercise 02
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Exercises on General Relativity and Cosmology

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–Home Exercises–
Due 27 April 2011

Exercise 02.1: Hydrodynamic Energy-Momentum Tensor ***(9 credits)***

A comoving observer in a *perfect fluid* will, by definition, see his surroundings as isotropic. In this frame, the energy-momentum tensor will be:

$$\tilde{T}^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

where ρ is the density and p the pressure of the fluid.

(a) Calculate the energy-momentum tensor $T^{\mu\nu}$ for an observer at rest. Assume the comoving observer's velocity to be $\vec{v}$. (*3 credits*)

(b) Show that $T^{\mu\nu}$ can also be written as

$$T^{\mu\nu} = (p + \rho)U^\mu U^\nu + p\eta^{\mu\nu} \,,$$

where U^μ is the four-velocity of the fluid. (*2 credits*)

(c) Consider an ideal gas (point particles that only interact in local collisions). Its energy-momentum tensor is:

$$T^{\mu\nu} = \sum_N \frac{p_N^\mu p_N^\nu}{E_N} \delta^3(\vec{x} - \vec{x}_N)$$

where E_N is the energy of N^{th} particle. Calculate the density ρ and pressure p for a comoving observer. (*2 credits*)

(d) If the particle number density, n, is defined as

$$n \equiv \sum_N \delta^3(\vec{x} - \vec{x}_N) \,,$$

what is the relation between ρ and p for a (*2 credits*)

(i) cool, non-relativistic gas

(ii) hot, *extremely* relativistic gas

Exercise 02.2: Coordinate charts for manifolds ***(2 credits)***

(a) Argue why the circle manifold, $\mathbb{S}^1$, cannot be covered by a single coordinate chart. Provide charts for this manifold. *(1 credit)*

(b) Can $\mathbb{R} \times \mathbb{S}^1$ be covered by a single chart? Provide a chart/charts for this manifold too. *(1 credit)*

Exercise 02.3: Conformal manifolds ***(4 credits)***

A *conformal manifold* is one which is equipped with an equivalence class of metrics with two metrics being equivalent if they differ by a smooth positive factor. Such an equivalence relation preserves angles but not lengths on the manifold. A manifold is *conformally flat* if, at the local neighborhood of every point, the metric is *conformally equivalent* to a flat metric ie. $g^{\mu\nu} = f(x^\mu)g^{\mu\nu}_{(0)}$ (where, $g_{(0)}$ is a flat metric, like Minkowski, Euclidean etc and f is a smooth positive function).

(a) Consider a Euclidean metric on a 2-sphere with $ds^2 = d\theta^2 + sin^2\theta\ d\phi^2$. Find a coordinate system where this metric is conformally flat. *(2 credits)*

(b) Show that a manifold in three or more dimensions is not, in general, conformally flat. (Hint: Compare the number of independent variable and the number of constraint equations) *(2 credits)*

Exercise 02.4: Effective potential in Newtonian gravity ***(5 credits)***

Consider a small body of mass m moving around a heavy (hence, stationary) body of mass M at a distance R

(a) Derive an effective potential for the motion of the body of mass m. (Hint: Write down the expression for the two constants of motion - Energy (KE + gravitational) and angular momentum. Use these to eliminate the angular coordinate and rearrange to get a form $V_{\text{eff}} = E_{\text{total}} - KE$) *(2 credits)*

(b) Show that, if its velocity is $\sqrt{2GM/R}$, the body of mass m will escape to infinity, irrespective of its initial direction, unless it is moving directly towards the center of the heavy body. *(3 credits)*

Remark: As you will see later, this fact that the particle escapes independent of its initial direction does not hold in GR. Close to the horizon of a black hole, a particle must move almost directly outwards in order to escape.