Physikalisches Institut
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Theoretische Physik

Exercise 09
8 June 2011
SS 2011

Exercises on General Relativity and Cosmology

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–Home Exercises–
Due 22 June 2011

Exercise 9.1: Hydrodynamics ***(20 credits)***

(a) Write the first law of thermodynamics for a relativistic fluid (ie. write the law of conservation of mass-energy for a fluid element). Taking the baryon number to be conserved, rewrite the first law in terms of baryon number density, n, and entropy per baryon, s, by eliminating volume, as follows: (*2 credits*)

$$d\left(\frac{\rho}{n}\right) = -Pd\left(\frac{1}{n}\right) + Tds$$

$$\text{ie. } d\rho = (\rho + P)\frac{dn}{n} + nTds$$

(b) Consider the energy-momentum tensor of a perfect fluid (*cf.* ex-2.1(b)). Use the equations of motion ($T^{\mu\nu}{}_{;\nu} = 0$) to show that the flow of a perfect fluid is isentropic (ie. $ds/dt = 0$) (*3 credits*)
Hint: Here and in the following, use: $g^{\mu\nu}{}_{;\nu} = 0$, $U^\mu = \{1, 0, 0, 0\}$ in rest frame and $U_\alpha U^\alpha{}_{;\nu} = \frac{1}{2}(U_\alpha U^\alpha)_{;\nu} = 0$. Also, use the conservation of number-flux vector of baryons ie. $(nU^\nu)_{;\nu} = 0$

(c) For a perfect fluid, show that the trace of the stress-energy tensor is negative if and only if: (*2 credits*)

$$\frac{d(\log\, \rho)}{d(\log\, n)} < 4/3$$

(d) An idealized description of heat flow in a fluid uses the heat flux four-vector, q, with components in the fluid rest frame as $q^0 = 0$ and q^j =(energy per unit time crossing a unit surface perpendicular to e_j, in the positive j direction). The stress-energy tensor associated with the heat flow is:

$$T^{\alpha\beta}_{\text{heat}} = U^\alpha q^\beta - U^\beta U^\alpha$$

Let s, n and q be the entropy per baryon, number density of baryons and heat flux respectively - all measured in the proper frame of the fluid. The entropy density-flux 4-vector, S, is:

$$S = nsU + \frac{q}{T}$$

where, U is the 4-velocity of the fluid rest frame. Consider a fluid that is "perfect" *except* for admitting some heat conduction, described by the heat flow 4-vector, q, above. Show that the local rate of entropy generation is:

$$\nabla \cdot S = -\frac{q \cdot a}{T} - \frac{\nabla T}{T^2} \cdot q$$

where, a is the 4-acceleration of the fluid: $a_\alpha = u_{\alpha;\beta} u^\beta$ (*4 credits*)
Hint: Use $\hat{T}^{\mu\nu}{}_{;\nu} U_\mu = 0$, where, $\hat{T}^{\mu\nu} = T^{\alpha\beta}_{\text{fluid}} + T^{\alpha\beta}_{\text{heat}}$. Also use $q^\alpha U_\alpha = 0$ (why?) and $(nU^\nu)_{;\nu} = 0$.

(e) In a uniformly accelerating system, show that the condition for thermal equilibrium is *not* T=constant, but rather is:

$$T = T_0 \exp(-a \cdot x)$$

where, x is the coordinate position in the accelerating frame. (*2 credits*)
Hint: At thermal equilibrium, $\nabla \cdot S = 0$

(f) Now consider a viscous fluid. If u is the 4-velocity of the viscous fluid, show that ∇u can be decomposed as:

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta$$

where, a is the 4-acceleration of the fluid: $a_\alpha = u_{\alpha;\beta} u^\beta$
θ is the *expansion* or *divergence* of the fluid worldlines:

$$\theta = \nabla \cdot u = u^\alpha{}_{;\alpha}$$

$\omega_{\alpha\beta}$ is the *rotation 2-form* or *vorticity* of the fluid:

$$\omega_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha;\mu} P^\mu{}_\beta - u_{\beta;\mu} P^\mu{}_\alpha\right)$$

$\sigma_{\alpha\beta}$ is the *shear tensor*:

$$\sigma_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha;\mu} P^\mu{}_\beta + u_{\beta;\mu} P^\mu{}_\alpha\right) - \frac{1}{3}\theta P_{\alpha\beta}$$

Here, P is the *projection tensor* that projects a vector onto the 3-surface perpendicular to u:

$$P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$$

(*3 credits*)

(g) The energy-momentum tensor of an imperfect fluid is:

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + p P^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} - \zeta\theta P^{\alpha\beta}$$

Here, η and ζ are respectively the coefficients of shear and bulk viscosity. Show that the viscous terms lead to the production of entropy at the rate: (*4 credits*)

$$S^\alpha{}_{;\alpha} = \left(\zeta\theta^2 + 2\eta\sigma_{\alpha\beta}\sigma^{\alpha\beta}\right)/T$$

Hint: Use $(T^{\alpha\beta} U_\alpha)_{;\beta}$, $U_{\alpha;\beta}$ from 9.1(f), $(nU^\nu)_{;\nu} = 0$ and 1st law from 9.1(a).
Remark: The equations of motion (ie. $T^{\alpha\beta}{}_{;\beta} = 0$) reduce to the Navier-Stokes equations in the non-relativistic limit.