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| Physikalisches Institut | Exercise 0 |
| Universität Bonn | April 3rd |
| Theoretische Physik | Summer Semester 2019 |

# Exercises on General Relativity and Cosmology

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`http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html`

## –Opening Remarks–

The lecture is accompanied by exercise sessions of two hours per week, which will start next week already. The exercise sheets will usually be handed out on Wednesdays, during the lecture, and are due on the following Wednesday. For admission to the final exam, you need to achieve at least 50% of the total points from all exercise sheets. Your grade will then be based upon performance in the final exam; i.e. the exercises do not contribute to your final mark.

Latest information will be posted on the webpage. The exercise sheets will also be available on the webpage, so you can help protect the environment :).

Bardia will be preparing the sheets and organizing the exercise classes. He's reachable at

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There will be four tutorial groups, as we agreed in the lecture. Here is the info:

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| Amitayus Banik | Mondays | 14 - 16 | SR II – HISKP |
| Benoit Scholtes | Tuesdays | 10 - 12 | Raum 0.021 – AVZ I |
| Gianluca Stellin | Thursdays | 10 - 12 | Konfrenzraum I – PI |
| Daniel Galviz | Fridays | 12 - 14 | Konfrenzraum I – PI*** |

*** On April 12th and May 10th, the Friday Tutorial will be held in Seminar Room 1, BCTP.

## –Class Exercises–

April $8^{\text{TH}}$ - $12^{\text{TH}}$

**C.1 Spacetime Diagrams**

In the following exercise we consider, for simplicity, a two(temporal+spatial) dimensional spacetime.

(a) Draw a spacetime diagram $(x, t)$ and present the following objects

  (i) an event.

  (ii) a light-ray.

  (iii) the worldline of an object that travels with velocity $v < 1$.

  (iv) the worldline of an object that travels with velocity $v > 1$.

  (v) the worldline of an accelerated object.

(b) Draw a spacetime diagram $(x, t)$ of an observer $\mathcal{O}$ at rest. Into this spacetime diagram draw the worldline of an observer $\mathcal{O}'$ that travels with velocity $v$ measured in the restframe of $\mathcal{O}$. What are the coordinate axes of the spacetime diagram of $\mathcal{O}'$?
*Hint: What is his time-axis? How do you then construct the space-axis?*

(c) Remember the relativistic *length contraction* phenomenon where a length $l'$ in the frame of the observer $\mathcal{O}'$ appears as a length $l$ to the observer $\mathcal{O}$, with

$$l = \sqrt{1 - v^2} l' .$$

In the following we consider the so-called garage *paradox*. We consider a car and a garage that have both length $l$ at rest. The garage has a front (F) and a back (B) door. It is constructed in such a way, that it opens both doors when the front of the car arrives at the front door, closes both doors, if the back of the car reaches the front-door and opens both doors again, when the car starts leaving the garage (ie. the front of the car arrives at the back-door). From the point of view of the garage the car is length-contracted and nicely fits into the garage. From the point of view of the car, though, the garage is length-contracted and the car will not fit, but instead will be destroyed by the doors. What *really* happens?
*Hint: Draw a spacetime diagram in which the garage is at rest. What is the order in which the events appear for both observers?*

Let us assume that in our two dimensional spacetime, an observer sees three events, A-B and C, in the order $ABC$; There exists a second observer that sees them in order $CBA$.

d) In two-dimensional (1+1) Minkowski space, can there be a third observer for whom the events appear in order $ACB$? Argue for your answer, for example by drawing a spacetime diagram.

e) Does this carry over to higher-dimensional Minkowski space, let's say our beloved 4d?

### C.2 The Lorentz group

We consider four-dimensional Minkowski space $\mathbb{R}^{1,3}$, which is $\mathbb{R}^4$ equipped with the Minkowski metric $\eta$. This is a symmetric, non-degenerate bilinear form $\eta : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ defined by

$$\eta(e_\mu, e_\nu) \equiv \eta_{\mu\nu} = \begin{cases} -1 & \text{for } \mu = \nu = 0 \\ +1 & \text{for } \mu = \nu = 1,2,3 \end{cases} \tag{1}$$

for the standard orthonormal basis $\{e_0, e_1, e_2, e_3\}$ on $\mathbb{R}^4$. Using linearity we then find

$$\eta(x,y) = x^{\mathrm{t}} \cdot \tilde{\eta} \cdot y \quad \text{for } x, y \in \mathbb{R}^{1,3}, \tag{2}$$

where $\tilde{\eta}$ is a matrix with entries $\eta_{\mu\nu}$. From now, we identify $\tilde{\eta}$ and $\eta$ with each other and do not distinguish between them.

For $x, y \in \mathbb{R}^{1,3}$ we write $x \cdot y = \eta(x,y)$ and $x^2 = x \cdot x$. The postulates of special relativity imply that transformations $\Lambda$ relating two inertial frames, so called Lorentz transformations, preserve the spacetime distance, i.e.

$$(x-y)^2 = (\Lambda(x-y))^2 \quad \text{for all} \quad x, y \in \mathbb{R}^{1,3}\,. \tag{3}$$

This leads to the definition of the *Lorentz group*

$$\mathrm{O}(1,3) = \{\Lambda \in \mathrm{GL}(4,\mathbb{R}) \,|\, \Lambda^{\mathrm{t}} \eta \Lambda = \eta\}\,. \tag{4}$$

a) Show that $\Lambda \in \mathrm{O}(1,3)$ indeed fulfills eq. (3).

b) Show that $\mathrm{O}(1,3)$ indeed is a group.

c) Show that $\Lambda^{\mathrm{t}}\eta\Lambda = \eta$ written in components reads $\eta_{\rho\sigma}\,\Lambda^\rho{}_\mu\,\Lambda^\sigma{}_\nu = \eta_{\mu\nu}$.

d) Embed the group of three-dimensional rotations into $\mathrm{O}(1,3)$.

e) Show that $|\Lambda^0{}_0| \geq 1$ and that $|\det\Lambda| = 1$. With this argue that the Lorentz group consists of four branches (which are not continuously connected to each other). Hint: Use $\det(\mathbb{1} + \epsilon\lambda) = 1 + \epsilon\,\mathrm{tr}\,\lambda + \mathcal{O}(\epsilon^2)$.

f) Show that the subset $\mathrm{SO}^+(1,3) = \{\Lambda \in \mathrm{O}(1,3) \,|\, \det\Lambda = 1\,,\, \Lambda^0{}_0 \geq 1\}$ forms a subgroup of $\mathrm{O}(1,3)$, called the *proper orthochronous Lorentz group*.

g) Identify the Lorentz transformations for time and parity reversal and relate them to the respective branches.

Consider two inertial frames, $K$ and $K'$. When $K'$ moves in $K$ with velocity $v$ in positive $x_1$ direction, the Lorentz transformation from $K$ to $K'$ is ($c = 1$)

$$\Lambda_{x_1}(v) = \begin{pmatrix} \gamma & -\gamma\cdot v & 0 & 0 \\ -\gamma\cdot v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

Transformation of this type are called *boosts*. We introduce the *rapidity* $\phi$ by $v = \tanh\,\phi$.

h) Rewrite $\Lambda_{x_1}(v)$ from eq. (5) in terms of the rapidity.

i) Consider two succsessive boost, both in the $x_1$ direction but with different velocities. Find the rapidity of the composite boost. Deduce the relativistic rule for addition of velocities.