Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 1
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

–Exercise One–

Due on April 10

H 1 Field Theory: The Example of Classical Electrodynamics *(15 points)*

In this exercise we consider the field theoretical formulation of classical electrodynamics. The electromagnetic field is described in terms of a vector field

$$A : \mathbb{R}^{1,3} \longrightarrow \mathbb{R}^{1,3}, \tag{1}$$

with components A^μ. From these components, we can build the field strength tensor F, via

$$F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu. \tag{2}$$

The particles of mass m_i and charge q_i are described by their trajectories,

$$x_i : I_k \subset \mathbb{R} \longrightarrow \mathbb{R}^{1,3} \quad \text{for } i = 1, ..., N = \text{number of particles} \tag{3}$$

$$\sigma_i \longmapsto x_i(\sigma_i), \tag{4}$$

For each particle, then σ is an arbitrary parameter used to parameterize its trajectory, a specific point on the trajectory curve then corresponds to coordinates x^μ. Written in coordinates, the total action is

$$S[x_i, A] = -\sum_{i=1}^{N} \int_{I_k} \mathrm{d}\sigma_i \left(m_i \sqrt{-\eta_{\alpha\beta} \dot{x}_i{}^\alpha(\sigma_i) \dot{x}_i{}^\beta(\sigma_i)} - q_i\, A_\alpha(x_i(\sigma_i)) \dot{x}_i{}^\alpha \right) \tag{5}$$

$$- \frac{1}{4} \int_{\mathbb{R}^{1,3}} \mathrm{d}^4 x\, F_{\alpha\beta}(x) F^{\alpha\beta}(x), \tag{6}$$

where $\dot{x}_i = \frac{\mathrm{d}}{\mathrm{d}\sigma_i} x_i$ and integration on Minkowski space is the same as on $\mathbb{R}^4$.

a) Shortly comment on the significance of each term in S. (*1 point*)

b) Take the variation of S with respect to x_i^μ in order to derive the Einstein-Lorentz equation (*2 points*)

$$m_i \frac{\mathrm{d}}{\mathrm{d}\sigma_i} \frac{\dot{x}_i{}^\mu(\sigma_i)}{\sqrt{-\eta_{\alpha\beta} \dot{x}_i{}^\alpha(\sigma_i) \dot{x}_i{}^\beta(\sigma_i)}} = q_i\, F^\mu{}_\nu(x_i(\sigma_i)) \dot{x}_i{}^\nu. \tag{7}$$

c) Rewrite the second term in S, the term where q_i appears, in terms of the charge-current density (*1 point*)

$$j^\mu(x) = \sum_{i=1}^{N} q_i \int \mathrm{d}\sigma_i \, \delta^{(4)}(x - x_i(\sigma_i)) \, \dot{x}_i{}^\mu(\sigma_i). \tag{8}$$

d) Take the variation of S with respect to A_μ to derive the inhomogenous Maxwell's equations (*2 points*)

$$\frac{\partial}{\partial x^\mu} F^{\nu\mu}(x) = j^\nu(x). \tag{9}$$

e) Use the definition of the field strength tensor, eq. (2), to show the homogenous Maxwell's equations (*2 points*)

$$\frac{\partial}{\partial x^\alpha} F_{\mu\nu} + \frac{\partial}{\partial x^\mu} F_{\nu\alpha} + \frac{\partial}{\partial x^\nu} F_{\alpha\mu} = 0. \tag{10}$$

Now take $A^\mu = (\phi, \vec{A})$ with ϕ and $\vec{A}$ such that $\vec{B} = \mathrm{rot}\, \vec{A}$ and $\vec{E} = -\,\mathrm{grad}\, \phi - \dot{\vec{A}}$. Further, $j^\mu = (\rho, \vec{j})$ with the charge-density ρ and current-density $\vec{j}$.

f) Express the components of the field strength tensor, $F_{\alpha\beta}$, in terms of the components of the electric and magnetic field, $\vec{E}$ and $\vec{B}$. (*1 point*)

g) Show that eq. (9) indeed gives the inhomogenous Maxwell's equations: (*2 points*)

$$\mathrm{div}\, \vec{E} = \rho, \qquad \mathrm{rot}\, \vec{B} - \dot{\vec{E}} = \vec{j}. \tag{11}$$

h) Show that eq. (10) indeed gives the homogenous Maxwell's equations: (*2 points*)

$$\mathrm{div}\, \vec{B} = 0, \qquad \mathrm{rot}\, \vec{E} + \dot{\vec{B}} = 0. \tag{12}$$

i) Parameterise eq. (7) by time, i.e. $\sigma = x^0 = t$, and show that it reduces to (*2 points*)

$$m \frac{\mathrm{d}}{\mathrm{d}t} \frac{\vec{v}}{\sqrt{1 - v^2}} = q \left(\vec{E} + \vec{v} \times \vec{B} \right). \tag{13}$$