Physikalisches Institut
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Theoretische Physik

Exercise 10
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

– Long Exercise Ten –

Due June 19^{th}

This sheet is gonna be (even ;))somewhat longer than the usual ones; you might not want to leave it off for the very end.

After developing the formalism and machinery we need in GR, and as you might expect, our job from here on will be applying this hard earned formalism to cases of special interest. You have already started out on that in the lecture by looking at the Schwarzschild solution for the metric; then you went on to apply that metric to the case of planetary motion. In this exercise we would like to work in the same direction; use our knowledge of GR to look at specific systems. We will be looking at three cases of different scales:

- An everyday, familiar satellite floating around the earth.
- A ray of light, passing an object of astrophysical size.
- And finally, gravitational waves propagating through spacetime.

We will only start on the last topic, and we'll pick up the rest on the next sheet.

H 10.1 Perks of living on the ISS (*8 points*)

A good approximation to the metric in the vicinity of the surface of the Earth is

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{1}$$

where

$$\Phi = -\frac{GM}{r} \tag{2}$$

may be thought of as the familiar Newtonian gravitational potential. Here G is Newton's constant and M is the mass of the earth. For this problem Φ may be assumed to be small.*(Can you then tell how you get to this metric from the Schwarzschild solution?)*

a) Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster? (*3 points*)

b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$? (*2 points*)

c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the earth, neglecting air resistance) completes one orbit? You can work to first order in Φ. Plug in the actual numbers for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface? (*3 points*)

H 10.2 *Deflection* of light (*12 points*)

In the lecture, you were introduced to the calculation of the deflection a freely propagating light ray, around a spherically symmetric mass. In this exercise, we would like to do the same calculation in more detail.

For a spherical symmetric and stationary mass distribution of mass M, the external background metric is chosen to be the Schwarzschild metric

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) , \tag{3}$$

where

$$A(r) = \left(1 - \frac{2GM}{r}\right) , \qquad B(r) = \left(1 - \frac{2GM}{r}\right)^{-1} ,$$

r is the distance to the center of mass.

a) Keeping $A(r)$ and $B(r)$ general for the moment, write down the geodesic equations. (*2 points*)

b) We can use the spherical symmetry to put $\theta = \frac{\pi}{2}$. Integrate the geodesic equations suitably to get

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{1}{A(r)} , \qquad r^2\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = J = \text{const.} , \qquad B(r)\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{J^2}{r^2} - \frac{1}{A(r)} = -E = \text{const.} ,$$

where λ is the parameter along the worldline.
(this should also remind you of Killing vectors.) (*2 points*)

c) Show that $\mathrm{d}\tau^2 = E\mathrm{d}\lambda^2$. What does this impose on the sign of E, if one considers photons or matter to travel on the geodesic respectively? (*1 point*)

d) Eliminate λ from the integrals of motion obtained in part (b) to obtain a direct relation between r and φ. Show that

$$\varphi = \pm \int \frac{\sqrt{B(r)}\mathrm{d}r}{r^2\sqrt{\frac{1}{A(r)J^2} - \frac{E}{J^2} - \frac{1}{r^2}}} . \tag{4}$$

(*2 points*)

Now consider a photon approaching the central mass from infinity with impact parameter b(see figure 1). Let r_0 be the minimum radial coordinate of the geodesic(minimum distance to the center of mass).

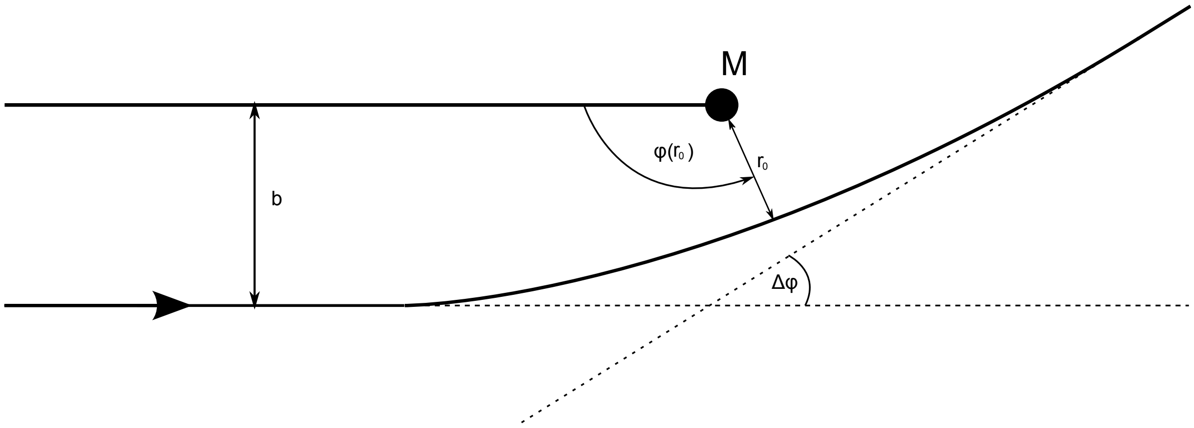

Figure 1: Deflection of a photon approaching a central mass with impact parameter b, $\Delta\varphi = 2\varphi(r_0) - \pi$.

e) Determine E and J in terms of r_0. (*1 point*)

f) Show that (4) reduces to

$$\varphi(r) = \int_r^\infty \frac{\sqrt{B(r')}}{\sqrt{\frac{r'^2}{r_0^2}\frac{A(r_0)}{A(r')} - 1}} \frac{\mathrm{d}r'}{r'} . \tag{5}$$

(*1 point*)

g) Use (5) and the approximations for $A(r)$ and $B(r)$ in the Newtonian limit, i.e. $2GM/r \ll 1$, to calculate the deflection angle $\Delta\varphi$. (*3 points*)
Hint: Show, that to lowest order in $2GM/r$,

$$\frac{r^2}{r_0^2}\frac{A(r_0)}{A(r)} - 1 = \left[\frac{r^2}{r_0^2} - 1\right]\left[1 - \frac{2GMr}{r_0(r+r_0)}\right] .$$

The following integrals may be useful

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{x} , \quad \int \frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x} , \quad \int \frac{\mathrm{d}x}{(x+a)\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a(x+a)} .$$

H 10.3 Gravitational waves in Vacuum (*20 points*)

In this exercise we will start getting away from the static solutions of GR; you've already learned in the lecture that the weak limit of static weak GR reproduces the Newtonian gravity. Here we would specifically like to look at the propagation of weak gravitational waves.

Before diving into the calculations, let us look ahead and think what we will have to do. The governing equation of motion is definitely the Einstein equations; non-linear and difficult to solve generically. But since we are interested in weak gravity regime, we will look for a *linearised* version, and try to deal with that.

Besides simplifying the equations by making them linear, there is another issue we expect we will need to address; gauge fixing. If you have had dealings with field theories before(electrodynamics is an example you have definitely seen.), you know that you will need to deal with the gauge freedom(or rather gauge *redundancies*) of the theory. We have basically designed the GR formalism with a huge deal of redundancies; we asked for invariance under general coordinate transformations; and at some point we will need to address that as well. To make a perturbative description of gravity, consider a coordinate system (U, x) of the spacetime manifold M, in which the metric g takes the form $g_{\mu\nu}(x)dx^\mu dx^\nu$ with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6}$$

where η denotes the Minkowski[1] flat metric. And let us limit our interest in a coordinate system where the weak gravity regime translates to $||h_{\mu\nu}|| \ll 1$.[2]. We would also like this smallness of h to be true in a region so that $||\partial_\sigma h_{\mu\nu}|| \ll 1$ as well.
We will then work up to the first order in h in our calculations.

a) To begin, show that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{7}$$

(*1 point*)

b) Show next, that the Christoffel symbols read

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \ . \tag{8}$$

(*1 point*)

c) Show then that to the order we are working in

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \ , \tag{9}$$

with $h = \eta^{\mu\nu}h_{\mu\nu} = h^\mu{}_\mu$, and $\Box$ being the D'Alembertian from flat space, $\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$. (*2 points*)

d) Show that this can be cast into

$$R_{\mu\nu} = \frac{1}{2}(\partial_\mu\partial^\lambda\bar{h}_{\lambda\nu} + \partial_\nu\partial^\lambda\bar{h}_{\lambda\mu} - \Box h_{\mu\nu}) \ , \tag{10}$$

with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$, being the *trace-reversed* perturbation. (*2 points*)

The next step is to deal with the *gauge* freedom corresponding to coordinate transformations. Assume we make a *nice* coordinate transformation, such that $x^\mu \to y^\mu(x) = x^\mu + \epsilon^\mu(x)$ with $||\epsilon^\mu|| \ll 1$ and $||\partial_\sigma\epsilon^\mu|| \ll 1$.

e) Start with the transformation of the metric g and show that

$$g_{\mu\nu}(y) = \eta_{\mu\nu} + h_{\mu\nu}(y) - \partial_\mu\epsilon_\nu(y) - \partial_\nu\epsilon_\mu(y). \tag{11}$$

(*2 points*)

[1] One can/should also consider cases where the *background* metric is not that of flat Minkowski; then you would be looking at the propagation of gravitational waves on a non-trivial background. Here, we consider the nicer case of flat background spacetime, as it is more relevant, and the vanishing of curvature components makes our calculations shorter and more relatable.

[2] This look like sort of limiting out choice of coordinate systems, but let us bear with that.

f) In the next step, show that equation 11 means that $\bar{h}$ transforms as

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - 2\partial_{(\mu}\epsilon_{\nu)} + \partial_\lambda \epsilon^\lambda \eta_{\mu\nu}. \tag{12}$$

(*2 points*)

g) We can now go back to part d) and notice that if

$$\partial^\mu \bar{h}_{\mu\nu} = 0, \tag{13}$$

there is a good deal of simplification in the Ricci tensor. This is an example of a gauge-fixing choice; called the *Lorenz gauge*. Fixing the gauge is equivalent to finding a specific coordinate transformation ϵ^μ. Show that for a given perturbation of the metric h, making a coordinate transformation by ϵ with

$$\Box \epsilon^\nu = \partial_\mu \bar{h}^{\mu\nu} \tag{14}$$

will put us in the Lorenz gauge. (*1 point*)

h) Once in the Lorenz gauge, show that the linearised Einstein equation reads

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{15}$$

(*2 points*)

At this point, we can specialize to the case of the vacuum solution; that is, we are not worrying about how a gravitational wave has come about, but rather would like to know how such an existing wave would propagate.

i) Verify that the plane wave

$$\bar{h}_{\mu\nu}(x) = a_{\mu\nu} \exp\left(i(k^\lambda x_\lambda + \phi)\right) \tag{16}$$

with constant k, constant symmetric a, and constant phase ϕ, is indeed a solution to the vacuum linearised Einstein equation, if k is light-like. (*2 points*)

The *polarization tensor*, is where the information about the configuration and degrees of freedom of gravitational waves lies. This should remind you of the u, v spinors in the Dirac theory, or the ϵ_μ polarization vectors in a vector boson theory if you've had dealings with those. Anyway, let us see what we know about a:

j) Remember that we are in the Lorenz gauge, so equation 13 holds. Show that this implies a *transverse* polarization tensor, i.e.

$$k_\mu a^{\mu\nu} = 0. \tag{17}$$

(*1 point*)

Is that it? is that all the constraints on the polarization? Remember that we got our first constraint on the polarization by imposing the Lorenz gauge. But there is still some fight left in the coordinate transformations. Looking back at part g), it is evident that if we make another coordinate transformation with parameters ϵ', with $\Box \epsilon'^\nu = 0$, we will still be in the Lorenz gauge.

k) Let us say that the coordinate transformation with $\epsilon'_\mu = b_\mu e^{ik_\sigma x^\sigma}$[3], with constant b, is one such transformation. Use what you know about the transformation of $\bar{h}$ from equation 12, to argue that this is equivalent to a transformation of the a tensor as

$$a_{\mu\nu} \to a_{\mu\nu} - ik_\mu b_\nu - ik_\nu b_\mu + i\eta_{\mu\nu} k_\lambda b^\lambda \ . \tag{18}$$

(4 points)

Now you have done enough algebra; so we can go a bit easier from here. It can be shown[4] that one can use this transformation to require that

$$a^\mu{}_\mu = 0 \texttt{ and } a_{0\nu} = 0. \tag{19}$$

Together, these are four new[5] constraints.[6]
We are now ready to finally do the physics. We started with a plane wave, with a symmetric polarization tensor a; that would mean we have $4(4+1)/2 = 10$ degrees of freedom to begin with. Imposing the Lorenz gauge in equation 17 kills four of these, leaving us with 6 dofs. The last bit of gymnastics in k) takes away four more, and then we are left with two for sure. You can play with these two degrees of freedom to define (left/right) or $(+/\times)$ polarizations for the wave[7].

[3]You need not worry about the complex parameter appearing.
[4]See for example Carroll's notes
[5]They might seem five, right? 1 from the trace condition + four from the $a_{0\nu}$, four of them together with the transverse condition we had before, imply the fifth; so there are four independent new constraints.
[6]Note that when we establish that $\bar{h} = a^\mu{}_\mu = 0$, then that means $h = -\bar{h} = 0$, and so that $h_{\mu\nu} = \bar{h}_{\mu\nu}$.
[7]It is nice to look at some of the animated gifs, visualizing the passing of a gravitational wave; there you can better relate to these two polarizations.