Physikalisches Institut
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Theoretische Physik

Exercise 11
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

– Exercise Eleven –

Due June 26th

This last exercise sheet contains three problems; the last two are bonus problems, put there to help in case someone needs some more points to be admitted to the exam. That, however, does **Not** mean they are irrelevant for, or out of the scope of the course or the exam.

H 11.1 Metric for the charged black hole ***(25 points)***

In the lecture you have found the Schwarzschild metric which is the vacuum solution to Einstein's equations for a spherically symmetric, static source. Here we generalize this discussion to the case in which the source is charged. So there is no more vacuum out there, but rather the EM field. It is still static and spherically symmetric, so we are more than welcome to make the same ansatz for the form of the metric as for the Schwarzschild solution,

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{1}$$

The non-vanishing components of the Ricci tensor are given by

$$\begin{aligned} R_{tt} =& e^{2(\alpha-\beta)}\left(2\frac{\alpha'}{r} + (\alpha')^2 - \alpha'\beta' + \alpha''\right)\,, \\ R_{rr} =& -(\alpha')^2 + 2\frac{\beta'}{r} + \alpha'\beta' - \alpha''\,, \\ R_{\theta\theta} =& e^{-2\beta}\left(-1 + e^{2\beta} - r\alpha' + r\beta'\right)\,, \\ R_{\phi\phi} =& R_{\theta\theta}\sin^2\theta\,. \end{aligned} \tag{2}$$

Due to the presence of charge, however, we now expect a non-vanishing electromagnetic field strength tensor $F_{\mu\nu}$. Let us make the ansatz

$$\begin{aligned} F_{tr} =& -F_{rt} = f(r)\,, \\ F_{\theta\phi} =& -F_{\phi\theta} = g(r)\sin\theta\,, \end{aligned} \tag{3}$$

with all other components vanishing.

a) Calculate the energy momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\,, \tag{4}$$

associated to the ansatz we made for the field strength tensor $F_{\mu\nu}$. Argue that this energy momentum tensor is indeed compatible with the spherically symmetric, static ansatz. *(4 points)*

b) Solve Maxwell's equations which for a curved spacetime read

$$g^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0\,,\, \nabla_{[\mu}F_{\nu\rho]} = 0\,, \tag{5}$$

to obtain

$$g(r) = c_1\,, \quad f(r) = e^{\alpha+\beta}\cdot\frac{c_2}{r^2}\,, \tag{6}$$

with constants c_1 and c_2.

(4 points)

c) Show that the energy momentum tensor calculated above is traceless.

(3 points)

With a traceless EMT, we thus have to solve

$$R_{\mu\nu} = \kappa T_{\mu\nu}\,. \tag{7}$$

d) Take a suitable linear combination of the tt and rr component of (7) to derive the relation

$$e^{2(\alpha+\beta)} = 1\,. \tag{8}$$

Note: Strictly speaking you will find $e^{2(\alpha+\beta)} = const.$ *This constant can again be rescaled to one without loss of generality.*

(2 points)

e) Consider the remaining equations to find the differential equations

$$\begin{aligned} 2e^{2\alpha}\left(\alpha'' + 2(\alpha')^2 + \frac{2}{r}\alpha'\right) - \frac{1}{4\pi}\kappa f^2 - \kappa\frac{g^2}{r^4} &= 0\,, \\ e^{2\alpha}\left(2r\alpha' + 1\right) - 1 + \frac{1}{4\pi}\frac{\kappa}{2r^2}(g^2 + f^2 r^4) &= 0\,. \end{aligned} \tag{9}$$

(4 points)

f) Solve the differential equations (9) to find the Reissner-Nordström metric

$$\begin{aligned} ds^2 &= -\Delta dt^2 + \Delta^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad \text{with} \\ \Delta &= 1 - \frac{c_3}{r} + \frac{1}{4\pi}\frac{\kappa}{2}\cdot\frac{c_1^2 + c_2^2}{r^2}\,, \end{aligned} \tag{10}$$

with another constant c_3.
Hint: The easy way would be to plug in the ansatz and determine c_1, c_2 and c_3. For another small simplification you can introduce $A = e^{2\alpha}$ and rewrite the differential equations in terms of A. *(4 points)*

g) Remember solving for the Schwarzschild metric; at this point, when we were left with constants in the solution, we looked at the weak-field limit, to relate the constants to a well-known parameter, the mass. In the same spirit, relate the integration constants c_1, c_2 and c_3 to the mass, electric charge and magnetic charge of the black hole. To this end analyse the asymptotics of the Reissner-Nordström metric and the electromagnetic field $F_{\mu\nu}$.

Hint:Note that the radial component of the magnetic field is given by $B^r = \epsilon^{01\mu\nu} F_{\mu\nu}$, with $\epsilon^{\rho\sigma\mu\nu} = \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\rho\sigma\mu\nu}$ as introduced early on in the lecture. For a magnetic monopole of charge p the radial component behaves as $B^r(r) = p/r^2$.

(4 points)

H 11.2 Kruskal-Szekres coordinates and apparent singularity *(25 **Bonus** points)*

In the lecture, the Schwarzschild solution was discussed along with some relevant phenomena, including the appearance of an event horizon; you saw that there was an *apparent singularity* as one approaches the Schwarzschild radius. Later, you dismissed the chances of anything *singular* happening at the Schwarzschild radius, and stated that the singularity was a mere coordinate effect(hence the name *coordinate singularity*) and that spacetime is in fact pretty *regular* at the Schwarzschild radius, if one chooses a better coordinate system. That is what we would like to do in this exercise; find a set of coordinates in which this apparent singularity is removed.

a) To start, show that the Schwarzschild metric in (11) does show a singularity at $r = 2GM$.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ . \tag{11}$$

b) Show that one can use the *tortoise*[1] coordinates(r^*) defined as[2]

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \ . \tag{12}$$

to bring the Schwarzschild metric to the form

$$ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2 \ , \tag{13}$$

[3] That is already rewarding, as it removes part of the funny behaviour of the metric we had before; namely, it keeps the light-cones from closing up in this new coordinate as is apparent from the form of the metric 13. But let us work further on; we would for one thing like to have the peculiar point $r = 2GM$ somewhere at hand in our coordinates, and not infinitely far away as in here with $r^* \to \infty$ as $r \to 2GM$.

c) Let us define the *Eddington-Finkelstein coordinates*, being *light-cone coordinates* constructed using the tortoise coordinate, as

$$\begin{aligned} \tilde{u} &= t + r^* \\ \tilde{v} &= t - r^* \ . \end{aligned} \tag{14}$$

Show then, that one can use these coordinates to write the metric as

$$ds^2 = \frac{1}{2}\left(1 - \frac{2GM}{r}\right)(d\tilde{u}d\tilde{v} + d\tilde{v}d\tilde{u}) + r^2 d\Omega^2 \ . \tag{15}$$

[1] Apparently after the thought experiment of *Achilles and the tortoise.*

[2] You need not worry about the the coordinates being defined for $r \geq 2GM$; as that is the region we will be involved with.

[3] The presence of r in the $d\Omega$ part is not the result of a typo, we simply decide to that part of the metric be, as it is pretty harmless; we all making all the effort to get rid of the apparent singularity in the other parts of the metric.

d) One last step is to introduce the *Kruskal-Szekres* coordinates, by

$$\begin{aligned} u' &= e^{\tilde{u}/4GM} \\ v' &= e^{-\tilde{v}/4GM} \quad , \end{aligned} \tag{16}$$

e) Show that one can express these new coordinates in terms of our original[4] coordinates, as

$$\begin{aligned} u' &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r+t)/4GM} \\ v' &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{(r-t)/4GM} \quad . \end{aligned} \tag{17}$$

f) Show that in terms of these latest new coordinates the metric becomes

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM}(du'dv' + dv'du') + r^2 d\Omega^2 \ . \tag{18}$$

[5]

H 11.3 Gravitational waves from a spinning rod (*10* ***Bonus*** *points*)

Consider a conducting metal rod of length L and mass density ρ, spinning with frequency ω.

a) Calculate the time-dependent part of the quadrupole moment q_{ij} and the luminosity

$$L = -\frac{1}{5}\left\langle \frac{d^3 q_{ij}}{dt^3}\frac{d^3 q^{ij}}{dt^3} \right\rangle \ . \tag{19}$$

(*5 points*)

Remember you had to start by finding the T^{00} element, and then calculate the moment from there.

b) Calculate the charge induced in the rod due to the centrifugal force. An order of magnitude approximation is sufficient. Will the rotation generate electromagnetic dipole radiation? (*3 points*)

c) Calculate the luminosity of the electromagnetic quadrupole radiation. What is the ratio between the power of electromagnetic and gravitational radiation for $\rho = 10\,\mathrm{g/cm^3}$ and $\omega = 1\,\mathrm{kHz}$?

(*2 points*)

[4] Do not get me wrong, there is nothing original about those coordinates, we just happened to start working with them when we started.

[5] Now all these changes of coordinates might seem random; I tried to keep the exercise short; you can look for a motivation for each of the changes of coordinates in for example Carroll or Hirata lecture notes for the Kruskal coordinates.