Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 2
April 10^{th}
Summer Semester 2019

Exercises on General Relativity and Cosmology

Dr. Stefan Förste, Bardia Najjari

http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

–Exercise Two :)–

Due on April 17^{th}

In the previous exercise, you started working with the Lorenz-invariant action of charged particles in presence of an electromagnetic field. From the action(eq. 5), and variating with respect to the proper fields, you found the equations of motion for the charged massive particle(eq. 7), and the electromagnetic field (eq. 9). Whereas for a single particle we are often interested in the momentum p^μ, in case of fields and *fluids*, it is rather useful to work with another object; the so called *Energy-Momentum Tensor*(EMT). As you saw partly in the lecture, the energy momentum tensor $T^{\mu\nu}$, roughly captures the p^μ *flowing through a surface of normal vector* $\hat{x}^\nu$. In this exercise sheet, we will familiarise ourselves with the energy momentum tensor, and look more closely at the EMT for electromagnetism.
Before we start, remember that in the lecture, you learned that for a scalar field ϕ, and it's corresponding Lagrangian

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

The *canonical* EMT is given by

$$T^\mu_\nu = \delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi$$

In fact, remembering how we got to this definition, you can tell that it is somewhat generic, so that instead of a scalar field ϕ, you can choose to use a component A^μ of a vector field, for example.

H 2.1 EMT for Pure Electromagnetism *(12 points)*

a) Use the definition, given above, to calculate the canonical EMT for a the field A^μ with Lagrangian

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) . \tag{0.1}$$

Notice that the current j^μ, introduced in the previous exercise, is now absent. *(2 points)*

b) As you saw in the lecture, this canonical EMT is not symmetric as a tensor. It is, however possible to remedy this, by adding to the canonical EMT, the term

$$\partial_\kappa (F^{\mu\kappa} A^\nu). \tag{0.2}$$

Choose the sign so that the result is a symmetric tensor. *(2 points)*

c) Calculate $\partial_\mu T^{\mu\nu}$, with the new EMT, and argue why this manipulation of the EMT was allowed. *(3 points)*

d) Explicitly show that the resulting EMT is *traceless*; that is

$$T^\mu_\mu = 0. \tag{0.3}$$

(2 points)

e) Calculate T^{00} and T^{0i}, in terms of the more familiar E, and B three-vectors, introduced in the previous sheet. Do these agree with our intuitive understanding of the EMT? *(3 points)*

H 2.2 Turning the Current Back On *(8 points)*

Let us turn the current j^μ back on, that is, add to our pure electromagnetism, a set of charged particles; that is we go back to the action given on the last sheet. The part of the Lagrangian, involving the field A, will then have an extra term

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu. \tag{0.4}$$

Now if we variate the corresponding action, <u>with respect to the field A</u>, the resulting EMT will also get a contribution due to the current j^μ.

a) Calculate the new symmetrised EMT corresponding to the field A. Let us call this tensor $\hat{T}$.
Hint: you don't really need to recalculate ;). *(1 point)*

b) Keeping in mind the presence of the current, so that $\partial_\mu F^{\nu\mu} = j^\nu$, calculate $\partial_\mu \hat{T}^{\mu\nu}$. How is this significant. *(3 points)*

c) Finally, note that the full EMT for a system is supposed to include the flow of momentum from every element that does carry it. In our case, this means that a complete form of the EMT, must include contributions from the charged particles as well.

In the lecture, you learned that for a particle of momentum p^μ, at coordinates $\tilde{x}$, the contribution to EMT is of the form

$$\Delta T^{\mu\nu} = \frac{p^\mu p^\nu}{p^0}\, \delta^{(3)}\left(\vec{x} - \vec{\tilde{x}}\right) \tag{0.5}$$

Show that if accounted for in the total EMT, such that

$$T_{\text{Total}} = \hat{T} + \sum \Delta T, \tag{0.6}$$

the problematic divergence vanishes once again; that is $\partial_\mu T^{\mu\nu}_{\text{Total}} = 0$. Could you have expected it? *(4 point)*