Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 3
April 17^{th}
Summer Semester 2019

Exercises on General Relativity and Cosmology

Dr. Stefan Förste, Bardia Najjari

http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

–Exercise Three–

Due April 24^{th}

H 3.1 EMT for a Perfect Fluid *(10 points)*

First, we would like to consider the energy-momentum tensor of a *perfect fluid*. A comoving observer will, by definition, see his surroundings as isotropic. In this frame the energy-momentum tensor is given by

$$\left(\tilde{T}^{\mu\nu}\right) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} ,$$

where ρ is the density and p the pressure of the fluid.

a) Calculate the components of energy-momentum tensor $T^{\mu\nu}$ for an observer at rest. Assume the comoving observer's velocity to be $\vec{v}$. (*3 points*)

b) Show that $T^{\mu\nu}$ can also be written as

$$T^{\mu\nu} = (p+\rho)U^\mu U^\nu + p\eta^{\mu\nu} ,$$

where U^μ are the components of the four-velocity of the fluid. (*2 points*)

c) From the nonrelativistic limit of the conservation of the energy momentum tensor, $\partial_\mu T^{\mu\nu}$, deduce *Euler's equations*

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 ,$$
$$\rho \left[\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = -\vec{\nabla} p .$$

Hint: The nonrelativistic limit is given by $(U^\mu) = (1, v^i)$, $|v^i| \ll 1$, $p \ll \rho$. Project the equation into pieces along and orthogonal to the four-velocity by contraction with U_ν and $P^\sigma{}_\nu = \delta^\sigma_\nu + U^\sigma U_\nu$ respectively. (*3 points*)

H 3.2 Getting used to Manifolds *(15 points)*

In the lecture the concept of manifolds has been introduced. This exercise is devoted to building up some intuition for these objects. To this end, let us first recall the definition: A differentiable manifold of dimension n is a topological space M, such that the space can be covered with a set of open sets $\{U_\alpha\}$ and for every α we have a diffeomorphism[1] $h_\alpha : U_\alpha \longrightarrow V_\alpha$ to an open set $V_\alpha \subset \mathbb{R}^n$. A pair (U_α, h_α) is called a chart, labelled α, and a set of charts covering M is called an atlas. Further, for every pair of charts(α, β) such that $U_\alpha \cap U_\beta \neq \emptyset$ the transition function $h_{\alpha\beta} \equiv h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \longrightarrow h_\alpha(U_\alpha \cap U_\beta)$ is required to be a diffeomorphism.

To get acquainted to this possibly new language, let us look at our jolly good $\mathbb{R}^n$ first.

a) Consider $\mathbb{R}^n$ itself as a topological space and explicitly construct an atlas to show that it is a manifold. (*1 point*)
Hint: One chart is enough, but feel free to try using more charts.

This result is not surprising, since manifolds are constructed such that they locally look like the $\mathbb{R}^n$; and of course, so does $\mathbb{R}^n$.

Let us go a step further. A manifold is called *topologically trivial* if it can be continuously shrunk to a point, an example is $\mathbb{R}^n$. One may think that this property of $\mathbb{R}^n$, being topologically trivial, allowed for covering it with one chart only.

b) Consider the infinitely long cyclinder M given by its *embedding* in $\mathbb{R}^3$,

$$M = \{(R\cos\phi, R\sin\phi, t) \,|\, \phi \in [0, 2\pi),\ t \in (-\infty, \infty),\ R > 0\} \ . \tag{1}$$

Although M is topologically non-trivial — it can be shrunk to a circle but not to a point — it can be covered with a single chart only. Construct such a chart explicitly. What is the dimension of this manifold, and how is this different from the dimension of the embedding space? (*3 points*)

Hint: Think about the punctured complex plane, $\mathbb{C} \setminus \{0\}$.

c) Now consider a torus given by its embedding in $\mathbb{R}^3$,

$$T = \{((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta) \,|\, \theta, \phi \in [0, 2\pi),\ R > r > 0\} \ . \tag{2}$$

Explicitly construct an atlas to show that T is a manifold. What is the dimension of this manifold? (*3 points*)

As many other objects, there is a notion of combining manifolds; the direct product of two manifolds, $M_{1,2}$ of dimensions $d_{1,1}$ is a manifold. Intuitively, the direct product amounts to putting an instance of manifold M_1 at every point on M_2.

d) Convince yourself(and preferably also your tutor) that the dimension of the resulting product manifold will be $d_1 + d_2$. (*1 point*)

[1] A diffeomorphism is a homeomorphism with the additional property that it and its inverse are continuously differentiable.

e) A nice example of this product might be writing the infinite cylinder we had from before as $M = \mathbb{R} \times S$, where S is just our familiar circle. The circle, is in fact a differentiable manifold. To be consistent, let us define the circle by its embedding in $\mathbb{R}^2$ as

$$S = \{(R\cos\phi, R\sin\phi)\,|\,\phi \in [0, 2\pi),\ R > 0\}\ . \tag{3}$$

What is the dimension of the circle as a manifold. Check that our assumption about the dimension of the product manifold works. Explicitly construct an atlas for the circle. Can you do this atlas with a single chart?
Hint: Remember that the keyword "open" in our definitions above. (*3 points*)

The lesson is that a higher dimension for the manifold, does not necessarily mean we need more charts in the atlas.
The feature that made the torus topologically non-trivial is called the *genus*, and the torus was of genus one. Another nice example, is provided by the sphere. The sphere, defined by its $\mathbb{R}^3$ embedding

$$Sph = \{(x, y, z)\,|x^2 + y^2 + z^2 = 1,\ x, y, z \in \mathbb{R}\}\ . \tag{4}$$

is a differentiable manifold of genus zero.

f) What is the dimension of this manifold? Explicitly construct an atlas to show that the sphere is indeed a manifold. How many charts, at least, do you need? Compare this with the case of the torus. (*4 points*)