Exercises on General Relativity and Cosmology

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–Exercise Four — In-class Exercise :)–

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C 4.1 Explicit Calculation on a Simple Manifold

Consider $\mathbb{R}^3$ as a manifold with flat Euclidean metric and coordinates $\{x, y, z\}$.

a) A particle moves along a parameterized curve given by

$$x(\lambda) = \cos\lambda\,, \quad y(\lambda) = \sin\lambda\,, \quad z(\lambda) = \lambda\,.$$

Express the path of the curve in spherical polar coordiantes.

b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

Now consider *prolate spheroidal coordinates*, which can be used to simplify the Kepler problem in classical mechanics. They are given by

$$\begin{aligned} x &= \sinh\chi \sin\theta \cos\phi \\ y &= \sinh\chi \sin\theta \sin\phi \\ z &= \cosh\chi \cos\theta\,. \end{aligned}$$

Consider the plane $y = 0$.

c) What is the coordinate transformation matrix $\frac{\partial x^\mu}{\partial x'^\nu}$ relating (x, z) to (χ, θ)?

d) What does the line element $\mathrm{d}s^2$ look like in prolate spheroidal coordinates?

C 4.2 Example of Induced Metric

In the lecture, you started learning about the metric tensor. A manifold M can be given an additional structure by endowing it with a metric tensor, which provides a natural generalization of the scalar product between two vectors in $\mathbb{R}^n$. In terms of the language of tangent and cotangent spaces, a metric tensor g is a smooth *tensor field* of type $(0, 2)$ on M that is symmetric and non-degenerate. This means:

- For every $p \in M$ there is a tensor g_p of type $(0, 2)$ on $V = T_pM$
- for every $u, v \in T_pM$ the equality $g_p(u, v) = g_p(v, u)$ holds
- if $g_p(u, v) = 0$ for every $v \in T_pM$ then $u = 0$

- the map $p \longmapsto g_p$ is continuous.

As in the case of the vectors, where you could write a vector in terms of its components, a tensor -and in this example the metric tensor- can be expanded in local coordinates $\{x^\mu\}$ on M as

$$g = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \tag{1}$$

in terms of smooth functions $g_{\mu\nu}$. A trivial example is that of $\mathbb{R}^n$ equipped with the standard euclidean metric, which is just what we call $\mathbb{R}^n$; another example is the same $\mathbb{R}^n$ manifold again, this time equipped with the Minkowski metric, resulting in the familiar n-dimensional Minkowski space $\mathbb{R}^{1,n-1}$. This shows that one and the same manifold can be equipped with different metrics, by which it is made into different objects.
Another important concept is that of the *pullback*. Consider a manifold M equipped with a metric g and a second manifold N. If we further have a smooth map $\phi : N \longrightarrow M$ we can use this map to pull g back onto N. This gives the so called *induced metric* on N, which is denoted as $\phi^* g$. With g as in eq. (1) and with local coordinates $\{y^\mu\}$ on N the induced metric locally reads

$$\phi^* g = \left[g_{\alpha\beta} \left(\frac{\partial \phi^\alpha}{\partial y^\mu} \right) \left(\frac{\partial \phi^\beta}{\partial y^\nu} \right) \right] \mathrm{d}y^\mu \otimes \mathrm{d}y^\nu \ . \tag{2}$$

a) Consider the two-sphere S^2 embedded in $\mathbb{R}^3$,

$$S^2 = \{ R(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \,|\, \phi \in [0, 2\pi),\ \theta \in [0,\pi),\ R > 0 \}. \tag{3}$$

Use the inclusion map we used for the embedding

$$\begin{aligned} \iota : \qquad S^2 &\longrightarrow \mathbb{R}^3 \\ (\theta,\phi) &\longmapsto R(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \end{aligned} \tag{4}$$

to calculate the induced metric on S^2.

b) do the same for the embedding of T^2 in $\mathbb{R}^3$. Use the corresponding inclusion map to calculate the induced metric on T^2. Remember the embedding was that of

$$T^2 = \{ ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta) \,|\, \theta, \phi \in [0, 2\pi),\ R > r > 0 \} \ . \tag{5}$$

c) In cosmology the so called *de Sitter space* will be of importance. This space is, can be recognized by its embedding, as cut out of five-dimensional Minkowski space $\mathbb{R}^{1,4}$ — with coordinates u, w, x, y, z, with u being the timelike coordinate— by the hyperboloid equation

$$-u^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2, \quad \alpha \in \mathbb{R} \ . \tag{6}$$

On de Sitter space we introduce coordinates t, χ, θ, ϕ and embed it in $\mathbb{R}^{1,4}$ by

$$\begin{aligned} u &= \alpha \sinh(t/\alpha) \\ w &= \alpha \cosh(t/\alpha) \cos\chi \\ x &= \alpha \cosh(t/\alpha) \sin\chi \cos\theta \\ y &= \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi \\ z &= \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi \ . \end{aligned} \tag{7}$$

Calculate the induced metric on de Sitter space.