Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

–Exercise Five :)–

Due May 8th

H 5.1 Vectors and Vector fields *(12 points)*

In the lecture, we introduced the directional derivatives along curves passing through a point P on the manifold, as the tangent space T_P. We would like to first confirm that the directional derivative does indeed form a vector space.
To that end; we should check that if $\frac{d}{d\lambda}$ and $\frac{d}{d\gamma}$ are two directional derivatives along two curves parameterized by λ and γ, then the combined operator $\mathcal{O} = a\frac{d}{d\lambda} + b\frac{d}{d\gamma}$, where a and b are constant numbers, is also gonna be in the same space. That is to say that $\mathcal{O}$ is a directional derivative. For an operator to be a well-defined derivative(*derivation*), it should satisfy

$$\begin{aligned} &\text{Linearity:} \quad && X(\alpha f + \beta g) = \alpha X(f) + \beta X(g) \quad && \text{with} \quad \alpha, \beta \in \mathbb{R},\ f, g \in C^\infty \\ &\text{Leibniz rule:} \quad && X(f \cdot g) = f \cdot X(g) + g \cdot X(f) \quad && \text{with} \quad f, g \in C^\infty\ . \end{aligned} \tag{1}$$

a) show by explicit calculation that $\mathcal{O}$ is a derivative. *(2 points)*

Let us move further with the vectors on a manifold. So far, we were working with the vector space defined at a point P on a manifold; pinning a vector space at each and every point on a manifold, we will have a *vector field*. Just as the vector space at point P was a map from functions to their derivative functions at the point P, a vector field maps functions to derivative functions, all over the manifold. A smooth vector field X on a manifold M fulfils the two above conditions, all over the manifold.
Given two vector fields X and Y we define a new vector field $[X, Y]$, the *Lie bracket* or *commutator* of X and Y, by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \text{for} \quad f \in C^\infty(M). \tag{2}$$

b) Show in two ways that $[X, Y]$ is indeed a vector field:

   i) Prove that $[X, Y]$ is a derivation.

   ii) Write $[X, Y]$ in terms of *components* and show that they transform as those of a vector field under change of coordinates.

(2 points)

c) What about XY or YX? are these combinations of vector fields, also vector fields themselves? *(2 points)*

d) Show that the Lie bracket is *(2 points)*

i) skew-symmetric, $[X, Y] = -[Y, X]$, and
ii) satisfies the Jacobi identity, $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.

e) Consider as the manifold, $\mathbb{R}^2$ equipped with some coordinates x^1, x^2. Calculate the Lie bracket of the coordinate vector fields $\partial_1 = \frac{\partial}{\partial x^1}$ and $\partial_2 = \frac{\partial}{\partial x^2}$. *(1 point)*

f) Find an example of two nowhere-vanishing, (at each point) linearly independent vector fields in $\mathbb{R}^2$ whose Lie bracket does not vanish. Note that these two vector fields provide a basis for the tangent space at each point. Can you find coordinates, under which this basis will be a coordinate basis? *(3 points)*

H 5.2 Tensors and (A)symmetry *(4 points)*

On the previous sheets, you already used the symmetry/anti-symmetry properties of tensors occasionally. We would like to do get a bit more used to tensors in that respect.
Let T be a tensor in the vector space $T^{0,k}(V)$. Given this tensor, one can then define the two related tensors.

$$T^{sym}_{\mu_1,\ldots,\mu_k} := T_{(\mu_1,\ldots,\mu_k)} = \frac{1}{k!} \sum_{\sigma \in S_K} T_{\mu_{\sigma_1},\ldots,\mu_{\sigma_k}} \tag{3}$$

is the corresponding symmetric tensor; you may be familiar with the () notation for symmetryzation, and S_k denotes the set of permutations of k objects. This will be a symmetric tensor, in the sense that, for example you dealt with a symmetric energy-momentum tensor on the previous sheets.
Similarly, there is another corresponding tensor

$$T^{Asym}_{\mu_1,\ldots,\mu_k} := T_{[\mu 1,\ldots,\mu_k]} = \frac{1}{k!} \sum_{\sigma \in S_K} sgn(\sigma)\, T_{\mu_{\sigma_1},\ldots,\mu_{\sigma_k}} \tag{4}$$

where again [ ] denotes antisymmetrization, implemented by the series on the right. $sgn(\sigma)$ is the sign, i.e. even/odd, of a specific permutation σ. This will be an antisymmetric tensor in the sense that the electromagnetic field strength tensor $F_{\mu\nu}$ was.

a) Let us specialize to the case $k = 2$, show that $T = T^{sym} + T^{Asym}$. *(1 point)*

This decomposition is often helpful, to see why, show that for two sets of tensors $X, Y \in T^{0,2}, T^{2,0}$. *(1 point)*

b) $X^{sym}\, Y^{Asym} = 0$.

c) $XY = X^{sym}Y^{sym} + X^{Asym}Y^{Asym}$

Let us now move on to $k = 3$.

d) can you decompose a tensor as $T^{Asym} + T^{sym}$? *(2 points)*