Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 6
May 8th
Summer Semester 2019

Exercises on General Relativity and Cosmology

Dr. Stefan Förste, Bardia Najjari

http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

–Exercise Six–

Due May 15th

H 6.1 Connections *(25 points)*

In the lecture you were introduced to the affine connection ∇ as a map

$$\begin{aligned}\nabla :\ \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M)\\ (X, Y) &\mapsto \nabla_X Y\,,\end{aligned}$$

with $\mathfrak{X}(M)$ being the space of vector fields on M. This mapping satisfies

$$\begin{aligned}\nabla_X(Y+Z) =& \nabla_X Y + \nabla_X Z\\ \nabla_{(X+Y)} Z =& \nabla_X Z + \nabla_Y Z\\ \nabla_{(fX)} Y =& f \nabla_X Y\\ \nabla_X(fY) =& X[f]Y + f\nabla_X Y\,,\end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$, and $f : M \to \mathbb{R}$ is a smooth function. The *connection components* ${\Gamma^\lambda}_{\nu\mu}$ are defined by[1]

$$\nabla_{\partial_\nu}\partial_\mu \equiv \nabla_\nu \partial_\mu = {\Gamma^\lambda}_{\nu\mu}\partial_\lambda\,.$$

Rewriting for two vector fields $X = X^\mu\partial_\mu$, $Y = Y^\mu\partial_\mu$,

$$\nabla_X Y = X^\mu \left(\frac{\partial Y^\lambda}{\partial x^\mu} + Y^\nu {\Gamma^\lambda}_{\mu\nu}\right)\partial_\lambda \equiv X^\mu \left(\nabla_\mu Y\right)^\lambda \partial_\lambda\,.$$

So far so well for vectors; now in order to define the action of the connection on general tensor fields, one first imposes the action of ∇_X on a function $f : M \to \mathbb{R}$ to be

$$\nabla_X f = X[f]$$

and then imposes the Leibniz rule, you were introduced to on previous sheets

$$\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1)\otimes T_2 + T_1 \otimes (\nabla_X T_2)\,,$$

where $X \in \mathfrak{X}(M)$ and T_1, T_2 are tensor fields of arbitrary types.

[1] As a reminder, the connection is not a tensor, and so the specifics of index placement on the object is a matter of convention and book keeping.

a) Use the above properties to argue

$$\nabla_\mu (V^\nu) = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda.$$

(*1 point*)

b) Let $\omega = \omega_\nu \mathrm{d}x^\nu$ be a one-form (a covector) and $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$ two vector fields. Use what we just asked of the connection and its action on functions, to derive the action of an affine connection ∇ on a covector ω,

$$(\nabla_X \omega)_\nu = X^\mu \partial_\mu \omega_\nu - X^\mu \Gamma^\lambda_{\mu\nu} \omega_\lambda$$

by looking at $\nabla_X (\omega Y)$.[2] (*2 points*)

It is easy to generalize this result to tensors of arbitrary type. Let T be a (q, r) tensor. Every upper index goes with a $+\Gamma$ and every lower one with a $-\Gamma$; that is

$$(\nabla_X T)^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} = X^\rho \partial_\rho T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} + X^\rho \Gamma^{\mu_1}{}_{\rho\kappa} T^{\kappa \mu_2 \dots \mu_q}{}_{\nu_1 \dots \nu_r} + \dots + X^\rho \Gamma^{\mu_q}{}_{\rho\kappa} T^{\mu_1 \dots \mu_{q-1} \kappa}{}_{\nu_1 \dots \nu_r}$$
$$- X^\rho \Gamma^\kappa{}_{\rho\nu_1} T^{\mu_1 \dots \mu_q}{}_{\kappa \nu_2 \dots \nu_r} - \dots - X^\rho \Gamma^\kappa{}_{\rho\nu_r} T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_{r-1} \kappa}.$$

Remember that just a few lines above, we defined the connection components, corresponding to a set of coordinates x^μ by

$$\nabla_\mu (V^\nu) = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda.$$

It should be obvious that we could have started from a different choice of coordinates y^α, such that[3][4]

$$\nabla_{\frac{\partial}{\partial y^\beta}} \left(\frac{\partial}{\partial y^\alpha} \right) = \nabla_{\partial_\beta} \partial_\alpha \equiv \nabla_\beta \partial_\alpha = \tilde{\Gamma}^\gamma_{\beta\alpha} \partial_\gamma \,.$$

and so

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \tilde{\Gamma}^\alpha_{\beta\gamma} V^\gamma$$

We are interested in tensors and tensorial transformation properties, so we would like to have

$$\nabla_\beta V^\alpha = \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^\mu} \nabla_\nu V^\mu.$$

c) Show that this means that connection components should transform as

$$\tilde{\Gamma}^\gamma_{\beta\alpha} = \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial y^\gamma}{\partial x^\lambda} \Gamma^\lambda_{\nu\mu} - \frac{\partial^2 y^\gamma}{\partial x^\mu \partial x^\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \,.$$

(*3 points*)

[2] You need to assume that the same Leibniz rule applies for the action of a vector on a one-form, or as Sean Carroll put it, assume that the connection "commutes with contractions".

[3] The tilde on Γ is there to distinguish from the connection components in the old coordinates; if that is not already clear.

[4] This is a somewhat sloppy notation, maybe you would like to think of all the new coordinate labels as primed, for example, $y^{\mu'}$; but that makes the equations look ugly and anyway it should be clear that we are labelling the two coordinates with $\alpha, \beta, \dots$ and $\mu, \nu, \dots$.

d) Show, by explicit calculation, that the above transformation rule for the connection components Γ, indeed makes $\nabla_X Y$ a vector. That is, explicitly show how $(\nabla_X Y)^\nu$ transforms. *(1.5 points)*

e) Show further, that with the same object Γ, the components of

$$(\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu}\omega_\lambda$$

transform as tensors, where $\omega = \omega_\nu \mathrm{d}x^\nu$ is a one-form field. *(1.5 points)*

f) Argue that for any two connections Γ and $\hat{\Gamma}$, the difference $\Gamma - \hat{\Gamma}$ transforms as a tensor, while neither of the two does so per se. *(1 point)*

This looks good, we now have a means of constructing a well-behaved derivative; it might, however, be worthwhile to look back for a second. Remember that $\partial_\mu V^\nu$ did not transform as a tensor. Also remember that we defined the vector U as a directional derivative $U = U^\nu \partial_\nu$.

g) We defined the Lie bracket of two vector fields $[V, U]$. Show once more, that $[V, U]$ does transform as a tensor. *(1 point)*

So far, by requiring the covariant derivative to behave tensorial, we have a very general connection object Γ. Now we demand that the metric $g_{\mu\nu}$ be *covariantly constant*, that is, if two vectors X and Y are parallel transported[5], then the inner product between them remains constant under parallel transport. The condition reads

$$(\nabla_\kappa g)_{\mu\nu} = 0\,.$$

If it satisfies this condition, the connection ∇ is said to be *metric compatible.*

h) Show that for a metric compatible connection ∇ with components $\Gamma^\lambda{}_{\mu\nu}$ the equation

$$\partial_\lambda g_{\mu\nu} - \Gamma^\kappa_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa_{\lambda\nu} g_{\kappa\mu} = 0$$

holds. Show that this implies[6]

$$\Gamma^\kappa_{(\mu\nu)} = \tilde{\Gamma}^\kappa_{\mu\nu} + \frac{1}{2}\left(T_\nu{}^\kappa{}_\mu + T_\mu{}^\kappa{}_\nu\right)\,,$$

where $\Gamma^\kappa_{(\mu\nu)} = \frac{1}{2}\left(\Gamma^\kappa{}_{\mu\nu} + \Gamma^\kappa{}_{\nu\mu}\right)$, $T^\kappa{}_{\lambda\mu} = 2\Gamma^\kappa{}_{[\lambda\mu]} = \Gamma^\kappa{}_{\lambda\mu} - \Gamma^\kappa{}_{\mu\lambda}$ and

$$\tilde{\Gamma}^\kappa_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)$$

are the Christoffel symbols.
Hint: Take a suitable linear combination of copies of the equation $(\nabla_\lambda g)_{\mu\nu} = 0$ with cyclic permutations of (λ, μ, ν). *(3 points)*

[5] If you have not covered parallel transport in the lecture, by the time you are reading this, do not worry, keep going.

[6] This $\tilde{\Gamma}$ is now not the coordinate transformed Γ or anything like that, it is just a name for this specific configuration of terms, known as the Christoffel symbols.

This implies, that the connection coefficients Γ are given by

$$\Gamma^{\kappa}{}_{\mu\nu} = \tilde{\Gamma}^{\kappa}_{\mu\nu} + K^{\kappa}{}_{\mu\nu} \,,$$

where

$$K^{\kappa}{}_{\mu\nu} \equiv \frac{1}{2}\left(T^{\kappa}{}_{\mu\nu} + T_{\mu}{}^{\kappa}{}_{\nu} + T_{\nu}{}^{\kappa}{}_{\mu}\right)$$

is called the *contorsion* as introduced in the lecture, whereas $T^{\kappa}{}_{\mu\nu}$ is called the *torsion tensor*.

i) If the connection on a manifold is symmetric in the lower indices, and so is so called *torsion-free*, show that the connection components Γ will be those of $\tilde{\Gamma}$, the Christoffel symbols. (*1 point*)

This torsion-free metric-compatible connection ∇, is called the Lev-Civita connection. We will be mainly concerned with this special connection in GR.

While the connection on a manifold was a very general object, admitting for the simultaneous existence of many connections, we saw that requiring for the connection to be torsion-free and metric compatible, pinned down the specific choice of Lev-Civita connection, which can be nicely calculated for a known metric.[7] Let us work out a few examples on that.

j) On the previous sheets you constructed the induced(pull-back) metric on the two-sphere S^2 and the torus T^2 embedded in $\mathbb{R}^3$ as well as de Sitter space embedded in $\mathbb{R}^{1,4}$. They were given respectively by

$$\begin{aligned} ds^2_{S^2} =& R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \\ ds^2_{T^2} =& r^2 d\theta^2 + (R + r\cos\theta)^2\, d\phi^2\,, \\ ds^2_{dS^4} =& -dt^2 + \alpha^2\cosh^2(t/\alpha)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]\,. \end{aligned}$$

for each case, calculate the Christoffel symbols. (*3+3+4=10 points*)

[7] Later we will see a somewhat more straight-forward method of calculating the Christoffel symbols, just so that you keep an open mind for that.