Physikalisches Institut
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Theoretische Physik

Exercise 7
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

–Exercise Seven–

Due May 22nd

H 7.1 Geodesic equation and the Christoffel symbols *(21 points)*

In the lecture we have seen that a curve is a geodesic if and only if there is a parametrisation such that it parallel transports its own tangent vector; that was true for any connection and thus an arbitrary parallel transport. In the special case in which the connection on the manifold is given by the Levi-Civita connection, which is the case we are concerned with in GR, given two points, a geodesic is also that curve c connecting the points, that *locally*[1] extremize the length functional

$$L(c) = \int_c \mathrm{d}s = \int_{\lambda_0}^{\lambda_1} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda\,, \tag{1}$$

where λ is the parameter of the curve. (Note that for simplicity we assume that $c \subset M$ is covered by a single chart.)

a) By varying the above functional, derive the geodesic equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = \frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\,, \tag{2}$$

where $e = \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}$.
Notice that Γ is now the Christoffel symbol, that is, the extremization singled out this specific choice for the connection. (*3 points*)

b) Show that if you parameterise the curve by its proper time τ the geodesic equation is simplified to

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = 0\,. \tag{3}$$

(*2 points*)

c) Remember when on the previous sheet, I mentioned in the last footnote that there is another method for calculating the Christoffel symbols? A tutor always pays his debts. You can look at the Lagrangian defined by $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ and use the generalized Euler Lagrange equation $\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial(dx^\alpha/d\lambda)} - \frac{\partial\mathcal{L}}{\partial x^\alpha} = 0$. Show that this leads to the same specific geodesic equation, and one can read the Christoffel symbols off it directly. (*3 points*)

[1] An nice example in Carroll's book is the case of a 2-sphere, given every two points that are not diagonally separated on the sphere, there is are two geodesics connecting them, i.e. the longer and the shorter part of the orthodrome(grand circle, but honestly orthodrome sounds cooler.) passing the two points, so you can immediately see that the longer path is a geodesic and so a local minimum of length, but globally there is this other geodesic that gives a shorter path.

Now, as an example of calculating the geodesic, let us consider geodesics of S^2 with metric $\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$.

c) Show, that the geodesic equations take the following form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = 0\,, \tag{4}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0\,, \tag{5}$$

where s is the arc length. (*1 point*)

d) Let $\theta = \theta(\varphi)$ be the equation of the geodesic. Show that the above equations can be written in one equation as follows

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\varphi^2} - 2\cot\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\varphi}\right)^2 - \sin\theta\cos\theta = 0\,. \tag{6}$$

(*1.5 points*)

e) Define $f(\theta) = \cot\theta$ and show that f fulfills the following differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\varphi^2} + f = 0\,. \tag{7}$$

What is the general solution? What do the geodesics of S^2 look like? (*2.5 points*)

As another example, we can look at the fictitious forces that are observed in non-inertial frames in Newtonian mechanics; they can be seen to arise from the metric connection. Given coordinates (t, x, y, z) in euclidean, 4-dimensional space-time let us consider the rotating coordinate system

$$t' = t, \quad x' = (x^2+y^2)^{\frac{1}{2}}\cos(\phi - \omega t), \quad y' = (x^2+y^2)^{\frac{1}{2}}\sin(\phi - \omega t), \quad z' = z, \quad \tan(\phi) = y/x\,. \tag{8}$$

f) Now we know the metric in the euclidean spacetime, and can do a change of coordinates to get the metric in new coordinates. Use the results form the previous parts of the exercise to calculate the equation of motion for a free particle in the non-inertial rotating coordinates given above. (*3 points*)

g) In the equation of motion, there is a part that does not fit the form of the geodesic equation we had before, rearrange if needed to separate these and identify the terms that describe the centrifugal and Coriolis forces (both fictitious) that arise in a rotating frame. This is an example of the geodesic equation in the presence of an external, here fictitious, force. But you can imagine you would need to add terms to the geodesic equation if for example the particle traversing the geodesic in your space-time is charged and subject to an electromagnetic field. (*3 points*)

h) Having the equation of motion, you should be able to read the Christoffel symbols, What about the Riemann Curvature tensor?(see next question.) (*2 points*)

H 7.2 Parallel transport and the curvature tensor *(9 points)*

The long wait is over and in the lecture you were introduced to a quantitative measure of curvature, the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$. It was given by the connection as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,. \tag{9}$$

a) What is the geometric interpretation of the Riemann curvature tensor? How is it defined at a point on the manifold while it has to do with moving vectors around on the manifold? Why does it have 4 indices? Is it (anti-)symmetric with respect to any indices, if so, what is the interpretation? *(1 point)*.

b) Let us try to understand this geometrical meaning a bit more carefully. Assume an infinitesimal *parallelogram* on the manifold[2] stretched along the two dimensions we choose to be 1, and 2. Let us refer to the four *corners*[3] of this parallelogram as A, B, C and D respectively, and say that for simplicity that in coordinates (x^1, x^2) these points will respectively be $(0,0)$, $(\delta a, 0)$, $(\delta a, \delta b)$, and $(0, \delta b)$.

- consider a generic vector V, originally belonging to the tangent space at point A; the vector will have components V^α.
- as a starting point, use the parallel transport equation to find $\frac{\partial V^\alpha}{\partial x^1}$.
- use the above differential equation to propagate V from A to B.
- similarly, parallel transport it along the path ABCDA, and find the resulting vector.
- show that if we call the resulting vector $\hat{V}^\alpha$, then

$$\left(\hat{V} - V\right)^\alpha = \delta a\, \delta b\, R^{\alpha}_{\lambda 12} V^\lambda \tag{10}$$

(4 points)

c) A closely related quantity is that of the commutator of two covariant derivatives along directions μ and ν. Instead of focusing on a vector V at a point P on the manifold and then pushing it around a loop, assume we have a vector field V; now let us think for a moment what $\nabla_\mu V$ means, in light of our recent knowledge of parallel transport. It is the deviation of the variation of vector V, along the curve with tangent vector in the direction μ, from the variation dictated by parallel transport. The commutator $[\nabla_\mu, \nabla_\nu]$ is then just moving first in the direction ν and then μ minus the the same operations with reversed order.
Show that

$$[\nabla_\mu, \nabla_\nu]\, V^\rho \;=\; R^{\rho}_{\sigma\mu\nu}\, V^\sigma - T^{\lambda}{}_{\mu\nu} \nabla_\lambda\, V^\rho \tag{11}$$

where as defined on the last sheet

$$T^{\kappa}{}_{\lambda\mu} = 2\Gamma^{\kappa}_{[\lambda\mu]} = \Gamma^{\kappa}_{\lambda\mu} - \Gamma^{\kappa}_{\mu\lambda} \tag{12}$$

(4 points)

[2] Do not be alarmed by the mention of a parallelogram on the manifold; it is true, we are no longer allowed to think of vectors on the manifold as stretching from one point to another, but you can imagine the corresponding parallelogram in the coordinates, and remember that manifolds locally look like $\mathcal{R}^n$.

[3] For simplicity, assume there is no torsion for the case you are interested in, so the parallelogram actually closes, your result will be however valid independent of the presence of torsion; if I got my math right, that effect is gonna be of order $(\delta a \delta b)^2$.