Physikalisches Institut
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Theoretische Physik

Exercise 8
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Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

–(: Exercise Eight :)–

Due May 29$^{\text{th}}$

H 8.1 Symmetries and Killing[1] Vectors *(30 points)*

Let us look back at what we have done so far, before we start with this exercise. Very early on, with experiments and thought experiments combined, we were convinced that we should let go of the comfortable life of working with a flat space-time, and get used to more general manifolds for a *curved* space-time. We then moved on to the mathematical formulation of the notion of manifolds, and reintroduced vectors, co-vectors, and tensors and fields of these objects, such as vector fields, etc. In another step forward we studied how we can move these objects around on the manifold, and take the well-behaving covariant derivative. That in turn, enabled us to quantify the notion of curvature, yielding the Riemann curvature tensor. This machinery is enough to address the problem we set out to solve, gravitation, and the motion of bodies in the presence of gravity/curvature, and that is what you will be doing in the lecture next.

Before moving on to gravitation, we would like to discuss symmetries; this will prove useful later on, but the idea should be kind of familiar: Just as in classical mechanics or special relativity, the presence of symmetries and their corresponding conserved quantities(e.g. momentum, angular momentum, etc.) greatly simplifies the analysis of problems.
What sort of symmetries are we talking about? First, remember that we usually spoke of building an atlas, or choosing a set of coordinates on a manifold; we also paid special attention to the transformation properties of different objects under a change of coordinates. We defined tensors as independent from the choice of coordinates, and transformed tensor components accordingly. All this work allows us to build our theory *coordinate independent* or *diffeomorphism invariant.* A second thing to remember is that we introduced a particularly important tensor, the metric. In this exercise, we will focus on *isometries*: the group of diffeomorphisms, that preserve the metric.
The Lie derivative acted on a tensor as:

$$\begin{aligned}\mathcal{L}_V T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} &= V^\sigma\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l}\\ &\quad -(\nabla_\lambda V^{\mu_1})T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} - (\nabla_\lambda V^{\mu_2})T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} - \cdots\\ &\quad +(\nabla_{\nu_1} V^{\lambda})T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} + (\nabla_{\nu_2} V^{\lambda})T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} + \cdots\,. \end{aligned} \tag{1}$$

where ∇_μ in our case represents the covariant derivative with the Levi-Civita connection.

[1] If I am not mistaken, you were introduced to Killing vectors last Monday, that is May 13th. Interestingly Wilhelm Killing was born on May 10th 1847 :).

a) Show that, as mentioned in the lecture, this means:

$$\mathcal{L}_V g_{\mu\nu} = 2\nabla_{(\mu} V_{\nu)}. \tag{2}$$

where the parentheses again represent symmetrization. (*1 point*)

b) show further that a vanishing Lie derivative of the metric, with respect to a Killing vector K means

$$g(\nabla_X K, Y) + g(\nabla_Y K, X) = 0. \tag{3}$$

for two vector fields X, and Y. (*1 point*)

So let us work out as an example, the Killing vectors for a 2-sphere with the metric pulled back from $\mathbb{R}^3$. These are

$$\begin{aligned} K_1 &= -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi\ , \\ K_2 &= +\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi\ , \\ K_3 &= \partial_\phi. \end{aligned} \tag{4}$$

Here, θ and ϕ are local coordinates of S^2, in terms of which the metric reads

$$g = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\ . \tag{5}$$

c) Explicitly show that these vectors satisfy the Killing equation. (*2 points*)

I mentioned that Killing vector fields can be used to simplify matters; below we will see the connection to conserved quantities and use them to find the geodesics on M.

d) Consider a geodesic $\mathcal{C}$ parameterized by λ, where λ is affinely related to the arc length. Show that for a Killing vector field K,

$$g\left(K, \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}\right) = \text{constant along the geodesic } \mathcal{C}\ . \tag{6}$$

Note that these are first order differential equations for the geodesic; as compared to the second order differential equation we had before. This should somehow remind you of what we had in classical mechanics for example; a second order equation of motion could be changed to a first order equation if using angular momentum conservation. (*3 points*)

e) We met the Lie bracket of vector fields $[X, Y]$ some time ago. Show that

$$[K_i, K_j] = -\epsilon_{ijk} K_k\ , \quad i, j, k = 1, 2, 3\ , \tag{7}$$

with the totally antisymmetric ϵ-tensor normalized as $\epsilon_{123} = 1$. (*3 points*)

f) The set of vector fields $K'_j = -iK_j$ for $j = 1, 2, 3$ and i just the imaginary unit, fulfil the angular momentum algebra. Can you think of a reason for this? (*2 points*)

g) For the three given Killing vectors, write the first order equation 6, in coordinates; label the constant appearing on the right hand side of the equation corresponding to $K = K_i$ as L_i. (*3 points*)

h) Combine the equations found in the previous item to arrive at an equation, in which θ, ϕ and the L_i but no derivatives of the geodesic coordinates appear. This equation can (at least locally) be solved to yield the geodesics in the form $\theta(\phi)$ or $\phi(\theta)$. (*2 points*)

i) Find the geodesics for the three cases in which only one L_i is non-zero. *(2 points)*

We got the geodesics, Hooray! but what would we have done if the Killing vectors where not given? Then we have to go and solve the Killing equation, which may or may have no easy solutions, or no solutions at all! as we saw that the Killing vectors corresponds to the amount of symmetry that the metric on the manifold possesses. In any case, "can't blame a physicist for trying"; we can use our intuition to guess some Killing vectors and make our lives easier with the first order equations of motion. Let us try this procedure the other way.

j) Consider the flat Euclidean metric in three dimensions; the good thing is that the Christoffel symbols are trivial, greatly reducing the complexity. Write down the Killing vector fields corresponding to rotations around the $x-$, $y-$ and $z-$axis. *(3 points)*

k) Transform these vector fields into spherical coordinates to derive the three Killing vector fields for the two-sphere. *(3 points)*

l) Find explicit expressions for a complete set of Killing vector fields for the Minkowski space, with metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. *(5 points)*
Hint: The Minkowski space is an example of maximally symmetric Lorenzian space-times, and you should be looking for $\frac{1}{2}4(4+1) = 10$ *Killing vectors.*