Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 9
May 29th
Summer Semester 2019

Exercises on General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/people/bardia/GRCss19/GR.html

– Exercise Nine –

Due June 6th

Tiny desk, BCTP

H 9.1 Back in the Elevator, Locally Inertial Frames (*20 points*)

Remember, that we started out with the Gedankenexperiment of being stuck in an elevator, with no knowledge of the outside world, and going about our usual business of doing physics. We then postulated the Einstein Equivalence principle as the idea that "In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field.". We are now in a position to see how we have actually formulated this idea, in the formalism we've developed so far.

It was argued in the lecture, by explicit counting, that it is always possible to locally find coordinates on M such that at the point p, $\partial_\sigma g_{\mu\nu}|_p = 0$. Hence the connection components vanish at that point. These coordinates are called *locally inertial coordinates.* Note that the second derivatives of the metric do not vanish in this coordinate system![1] Let us first remind ourselves how we got that conclusion.

Consider a Lorentzian manifold M with metric tensor g and a point $p \in M$. We start with general $g_{\mu\nu}(p)$ and without loss of generality we can assume that the coordinates of p are zero.

a) Argue that there are coordinates in which $g_{\mu\nu}(p) = \eta_{\mu\nu}$. (*2 points*)

b) Change coordinates from x^μ to $x'^\mu = x^\mu + b^\mu{}_{\alpha\beta}x^\alpha x^\beta$. Show that $g'_{\mu\nu}(p) = \eta_{\mu\nu}$ still holds and find $b^\mu{}_{\alpha\beta}$ such that the derivatives also vanish and $\partial'_\alpha g'_{\mu\nu}(p) = 0$. This implies that all Christoffel symbols vanish and we have constructed a locally inertial frame. (*3 points*)

c) Which coordinate transformations are we now still allowed to perform such that the transformed frame is stays locally inertial? (*2 points*)

d) Do the constructed coordinates always coincide with the *Riemann normal coordinates* introduced in the lecture? Provide arguments for your answer. (*2 points*)

So far so good. Now remember further, that on sheet 7 we formally introduced the Riemann curvature tensor. Let us spend some more time on the symmetries of the Riemann tensor, this will help get a better feeling of the geometrical significance of this object. In the following, it

[1] The metric at a point q near p can then be expanded as $g_{\mu\nu}(q) = \eta_{\mu\nu} + \frac{1}{3}R_{\mu\lambda\nu\rho}q^\lambda q^\rho + \dots$. Note that in this coordinate system p has coordinates $x = (0, \dots, 0)$. Note the appearance of the Riemann tensor, making the metric around point p non-trivial, could you have expected it?

is a good idea to use a local coordinate system to simplify some calculations. This is allowed, because if one finds a tensor equation, then (because of tensorial transformation behaviour under general coordinate transformations) it is true in every coordinate system.

e) Since we are interested in index symmetry properties, it is a good idea to consider the Riemann tensor with all indices lowered, $R_{\mu\alpha\beta\gamma} = g_{\mu\kappa} R^{\kappa}{}_{\alpha\beta\gamma}$. Use locally inertial coordinates to deduce the symmetry properties of the curvature tensor, i.e.

$$
\begin{aligned}
R_{\kappa\lambda\mu\nu} &= -R_{\kappa\lambda\nu\mu}\,, \\
R_{\kappa\lambda\mu\nu} &= -R_{\lambda\kappa\mu\nu}\,, \\
R_{\kappa\lambda\mu\nu} &= R_{\mu\nu\kappa\lambda}\,.
\end{aligned}
$$

(2 points)

f) Show, as was done in the lecture, that the sum of cyclic permutations of the last three indices of the curvature tensor vanishes, i.e.

$$R_{\kappa\lambda\mu\nu} + R_{\kappa\mu\nu\lambda} + R_{\kappa\nu\lambda\mu} = 0\,, \qquad 1^{\text{st}}\,\text{Bianchi identity}\,. \tag{1}$$

(1 point)

g) Use the results in (e)) to show that (4) is equivalent to the vanishing of the antisymmetric part of the last three indices of the Riemann tensor,

$$R_{\kappa[\mu\nu\lambda]} = 0\,.$$

(1 point)

h) Given these relationships between the different components of the Riemann tensor, how many independent quantities remain? Deduce the number of independent components of the Riemann tensor in n dimensions.[2] *(2 points)*

i) Make use of locally inertial coordinates once more to prove

$$\nabla_{[\mu} R_{\kappa\lambda]\rho\sigma} = 0\,, \qquad 2^{\text{nd}}\,\text{Bianchi identity}\,. \tag{2}$$

(3 points)

j) By contracting indices of the second Bianchi identity (5) twice, show that

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2}\nabla_{\nu} R\,.$$

(2 points)

H 9.2 Non-coordinate Basis and vielbeins *(5 points)*

When we started dealing with vector spaces on manifolds, we defined vectors at a point p on the manifold M by directional derivatives along curves passing through p. We then specialized to a set of curves corresponding to coordinates and the associated coordinate basis vectors.

[2] Look back at footnote 1, and remember that the metric in four dimensions had 20 non-vanishing double derivative degrees of freedom! How many degrees of freedom does the Riemann tensor have in 4 dimensions? Tada! :)

So far, we have almost exclusively worked with vectors in the coordinate basis. While we will probably not lose this habit up until the very end of this course, let us briefly take a look at the other scenario; non-coordinate basis. In the coordinate basis $T_p(M)$ is spanned by $\{\partial_\mu\}$ and $T_p^*(M)$ by $\{\mathrm{d}x^\mu\}$. You might remember that a coordinate basis was associated with vanishing Lie brackets. Now by definition, any given vector, can be expanded in terms of our coordinate basis vectors; and if a set of non-coordinates basis vectors exist, they will not be an exception. Let us label this new basis with $\alpha, \beta, \ldots$ and our jolly good coordinate basis vectors with the same old $\mu, \nu, \ldots$. Then we can in principle write

$$\hat{e}_\alpha = e_\alpha{}^\mu \partial_\mu \,, \qquad (e_\alpha{}^\mu) \in \mathrm{GL}(n, \mathbb{R}) \,,$$

where the $e_\alpha{}^\mu$ are just expansion coefficient matrices, such that $\det(e_\alpha{}^\mu) > 0$[3]. In addition we ask for our new basis vectors $\{\hat{e}_\alpha\}$ to be orthonormal with respect to $g_{\mu\nu}$, i.e.

$$g(\hat{e}_\alpha, \hat{e}_\beta) = e_\alpha{}^\mu e_\beta{}^\nu g_{\mu\nu} = \eta_{\alpha\beta} \,.$$

Where η is the canonical form of the metric we are dealing with(Minkowski or Euclidean in case of Euclidean or Lorenzian metrics). One last bit of notation; since this $e_\alpha{}^\mu$ was a matrix, let us denote the inverse of $e_\alpha{}^\mu$ by $e^\alpha{}_\mu$.

a) Show that the components of a vector V in the new basis $\hat{e}_\alpha$ are related to the old components V^μ by $V^\alpha = e^\alpha{}_\mu V^\mu$. *(1 point)*

b) Introduce the dual basis $\{\hat{\theta}^\alpha\}$ to $\{\hat{e}_\alpha\}$ by $\left\langle \hat{\theta}^\alpha, \hat{e}_\beta \right\rangle = \delta^\alpha_\beta$. Conclude that $\hat{\theta}^\alpha = e^\alpha{}_\mu \, \mathrm{d}x^\mu$. *(2 points)*

So now we have kind of established the existence of a non-coordinate basis and dual basis; $\{\hat{e}_\alpha\}$ and $\{\hat{\theta}^\alpha\}$. The objects relating the two sets of bases, i.e. the $e^\alpha{}_\mu$ are called the *vielbeins*.[4]

c) Show that the metric is identically given by $\mathrm{d}s^2 = \eta_{\alpha\beta}\, \hat{\theta}^\alpha \otimes \hat{\theta}^\beta$. *(1 point)*

d) Consider the standard induced metric on S^2 as you calculated in C 4.2. Calculate the non-coordinate basis $\hat{\theta}^\alpha$ as well as the *zweibeins* $e^\alpha{}_\mu$. *(1 point)*

The non-coordinate basis is of great interest in general relativity, because it allows for the definition spin connections and spinors on curved spacetimes. All that is beyond the scope of this lecture.

[3] We would like to keep the metric signature, for one thing.

[4] So ,in two dimensions that would be zweibeins, in four dimensions they are vierbeins, and the bigger numbers are out of the reach of my German knowledge ;).