Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 1
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Theoretical Particle Astrophysics

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http://www.th.physik.uni-bonn.de/people/berbig/SS20-TAPP/

Homework

Due May 5th, 2020

H.1.1 Quickies *(3 points)*

a) Briefly describe in your own words what is meant by a spatially isotropic and homogenous universe. (*1 point*)

b) State the definition of the Hubble parameter $H(t)$. What does the Hubble constant H_0 describe? (*1 point*)

c) The Hubble constant is ususally parameterized as $H_0 = h \times 100\,\mathrm{km\,Mpc^{-1}\,s^{-1}}$, where $h \approx 0.6 - 0.7$ depends on the exact measurement. Convert H_0 into natural units. (*1 point*)

H.1.2 Cutoff for high energy astro-physical neutrinos *(7 points)*

Last semester you encountered the Greisen–Zatsepin–Kuzmin (GZK) limit for the energy of cosmic ray protons. Protons with energies larger than about $5 \times 10^{19}\,\mathrm{eV}$ scatter of the Cosmic Microwave Background (CMB) photons via the formation of a Δ-resonance

$$p + \gamma_{\mathrm{CMB}} \to \Delta \to p + \pi^0 \quad \text{or} \quad n + \pi^+ \tag{1}$$

and loose too much energy (after one or multiple scattering processes) to reach earth. No cosmic ray protons with energies above this GZK cutoff ($5 \times 10^{19}\,\mathrm{eV}$) that travelled over a distance larger than the mean free path of this reaction (about 50 Mpc) have ever been observed on earth. Later during this lecture you will find out, that standard cosmology predicts the existence of a background of non-interacting relic (anti)-neutrinos called the Cosmic Neutrino Background (CνB). High energy neutrinos produced by astrophysical sources could scatter of relic anti-neutrinos via Z-boson exchange. The Standard Model cross section for this process

$$\nu + \overline{\nu} \to Z \to \text{fermions} \tag{2}$$

is only relevant around the Z-resonance at $s = m_Z^2$ and has a magnitude of about $\sigma_{\nu\overline{\nu}} = 1.5 \times 10^{-31}\,\mathrm{cm}^2$. In the following we assume a single generation of neutrinos with a mass of $m_\nu = 0.1\,\mathrm{eV}$.

a) Determine the energy E_ν an incoming neutrino needs to have in order to excite this resonance by scattering with a relic anti-neutrino, which you can assume to be at rest. *(1 point)*

b) Standard cosmology predicts that the CνB has a number density of $n_\nu = 55\,\text{cm}^{-3}$. Estimate the mean free path $l \approx (\sigma_{\nu\overline{\nu}}\, n_\nu)^{-1}$ in units of Mpc. *(2 points)*

c) Around the Z-pole the branching ratios (BRs) for the production of final state fermions are related to the Z decay BRs. In ca. 70% of cases the final state will be formed by hadronized quarks. Charged leptons make up roughly 10% of the possible final states and neutrinos belong to the remaining 20%. For the inelastic channels it is clear that the high energy neutrino flux will be depleted.[1]
For the elastic case we want to know how much energy the incoming neutrino will loose in the reaction $\nu(p_1) + \overline{\nu}(p_2) \to Z \to \nu(p_3) + \overline{\nu}(p_4)$. Assume the initial relic neutrino to be at rest and find an expression for the energy E_3 of the final state neutrino in terms of E_ν, m_ν and the scattering angle θ between the incoming and outgoing neutrino. What are the maximum and minimum values of E_3? Can the reaction (2) occur again for the outgoing neutrinos with the largest possible energy?
Hint: Look at $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ *with* $p_2 = \left(m_\nu, \vec{0}\right)^t$ *(2 points)*

d) Use your findings from the previous parts to argue whether or not there is a cutoff for the energy of cosmic neutrinos. If there is, compare the cutoff energy and the mean free path to the case for cosmic ray protons. *(2 points)*

H.1.3 Friedmann-Lemaitre-Robertson-Walker metric *(10 points)*

The line element for a spatially homogenous and isotropic expanding universe can be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \left[\frac{\mathrm{d}r^2}{1-\kappa\, r^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) \right] . \tag{3}$$

In this context $a(t)$ denotes the dimensionfull scale factor. κ paramterizes the spatial curvature. For $\kappa = 0$ we have a flat universe and $\kappa = 1\,(-1)$ describes a closed (open) geometry.

a) Write down the components of the metric tensor $g_{\mu\nu}$ for this line element. What are the components of the inverse metric $g^{\mu\nu}$? *(2 points)*

b) Show that the non-vanishing Christoffel symbols of the metric (3) are

$$\begin{aligned}
\Gamma^0_{11} &= \frac{a\,\dot{a}}{1-\kappa\, r^2}, & \Gamma^1_{11} &= \frac{\kappa\, r}{1-\kappa r^2}, \\
\Gamma^0_{22} &= a\,\dot{a}\, r^2, & \Gamma^0_{33} &= a\,\dot{a}\, r^2 \sin^2\theta, \\
\Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}, & \Gamma^1_{33} &= \sin^2\theta\, \Gamma^1_{22} = -r(1-\kappa\, r^2)\sin^2\theta\ , \\
\Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r}, & \Gamma^2_{33} &= -\sin^2\theta\, \Gamma^3_{23} = -\sin\theta\, \cos\theta\ ,
\end{aligned} \tag{4}$$

and those related to these by symmetry in the lower indices. We use the notation $(x_0, x_1, x_2, x_3) = (t, r, \theta, \phi)$. *(8 points)*

[1]Even if some of the final state hadrons and charged leptons decay back to neutrinos, the energies of the daughter neutrinos will be much smaller compared to the original neutrino.