Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 1
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Theoretical Particle Astrophysics

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http://www.th.physik.uni-bonn.de/people/berbig/SS20-TAPP/

Homework

Due May 12th, 2020

H.2.1 Quickies *(2 points)*

a) What is the definition of the cosmological redshift?. (*1 point*)

b) Explain the notion of co-moving coordinates. (*1 point*)

H.2.2 Equation of state for matter and radiation *(10 points)*

This exercise deals with statistical physics in the expanding universe. Our goal is to find the relationship between the pressure and energy density for a gas made up of matter or radiation. Each particle species has a distribution function $f(\vec{p})$, that depends on its energy $E(\vec{p})$, mass m and chemical potential μ as well as the temperature T. The distribution functions read

$$f(\vec{p}) = \frac{1}{(2\pi)^3} \frac{1}{\exp\left(\frac{E(\vec{p})-\mu}{T}\right) \mp 1}, \tag{1}$$

where the + (-) sign holds for fermions (bosons).

a) Matter is defined as a non-relativistic species that satisfies $m \gg E - \mu \gg T$. The distribution function can be approximated by a Maxwell-Boltzmann distribution

$$f(\vec{p})_{\mathrm{MB}} = \frac{1}{(2\pi)^3} \exp\left(-\frac{E(\vec{p})-\mu}{T}\right) \approx \frac{1}{(2\pi)^3} \exp\left(\frac{\mu - m}{T} - \frac{|\vec{p}|^2}{2m\,T}\right) \tag{2}$$

for both bosons and fermions. Compute the number density n of the non-relativistic particles starting from

$$n = g_i \int \mathrm{d}^3 p \; f(\vec{p}) \,. \tag{3}$$

In this context we label the number of spin degrees of freedom for a given particle species with g_i, which is 1 for a scalar, 2 for a Weyl-fermion and 2 (3) for a massless (massive) vector boson. For a gas of non-relativistic particles of mass m the energy density ρ can be approximated by $\rho \approx m\,n$.
Hint: For the evaluation of this and the following integrals you may consult an integral table or use a computer algebra program like Mathematica. (*2 points*)

b) Determine the pressure of the gas given by

$$P = \frac{4\pi}{3} g_i \int \mathrm{d}p \, \frac{p^4}{E(\vec{p})} f(\vec{p}) \,, \tag{4}$$

where $p = |\vec{p}|$. The factor of $1/3$ arises due to isotropy and 4π comes from the integration over the solid angle. For a non-relativistic gas it suffices to set $E(\vec{p}) \approx m$ in the denominator of the above. Use this to find the equation of state

$$\omega = \frac{P}{\rho}, \tag{5}$$

in the limit $T \ll m$. *(3 points)*

c) Now we consider a fermionic or bosonic gas of highly relativistic particles for which $T \gg E \gg m$ holds true. In cosmology such a gas is called radiation. For simplicity we neglect the chemical potential μ in the following. Determine the energy density

$$\rho = g_i \int \mathrm{d}^3 p \, E(\vec{p}) \, f(\vec{p}) \,, \tag{6}$$

for both a fermionic and a bosonic gas. *(3 points)*

d) Further determine the pressure (4) associated with these gases of radiation and compute the equation of state. *(2 points)*

H.2.3 Equation of state for an exotic scalar field *(8 points)*

In class you encountered the Lagrangian for a classical scalar field ϕ with a canonical kinetic term and a potential $V(\phi)$

$$\mathcal{L}_{\mathrm{sc}} = \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - V(\phi), \tag{7}$$

as an example of matter which lead to the following energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \, \mathcal{L}_{\mathrm{sc}}. \tag{8}$$

In this exercise we consider a **spatially homogenous** scalar field with the Lagrangian [1]

$$\mathcal{L} = -V_0 \sqrt{1 - g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi} \,, \quad \text{where} \quad V_0 > 0. \tag{9}$$

a) Find the components of the energy momentum tensor for this scalar field. Since this Lagrangian can not simply be separated into kinetic and potential terms like $\mathcal{L}_{\mathrm{sc}}$, we can not reuse the result in equation (8), but have to start from the definition of the energy-momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} \, \mathcal{L} \right)}{\delta g^{\mu\nu}}. \tag{10}$$

Here we used the notation $g = \det(g_{\mu\nu})$. *(4 points)*

b) Use your previous findings to compute the energy density ρ and the pressure P defined via

$$\rho = T_{00} \quad \text{and} \quad T_{ij} = -P \, g_{ij}. \tag{11}$$

Continue by determining the corresponding equation of state $\omega = P/\rho$. What happens for $\partial_t \phi \ll 1$? *(4 points)*

[1] The scalar field has been normalized appropriately so that $(\partial\phi)^2$ is dimensionless.