Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 8
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Theoretical Particle Astrophysics

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https://www.th.physik.uni-bonn.de/people/berbig/SS20-TAPP/

Homework

Due July 7th, 2020

H.8.1 Quickies *(4 points)*

a) Name two empirical motivations for the existence of Dark Matter (DM). (*2 points*)

b) State two properties a hypothetical DM particle needs to satisfy. Why do we call it 'dark' matter? (*2 points*)

H.8.2 Gravitino Dark Matter *(9 points)*

Supersymmetry (SUSY) is a well motivated framework for extending the Standard Model and adressing its formal and phenmenological shortcomings. Supersymmetry postulates a new kind of symmetry relating fermions and bosons and introduces superpartners for all Standard Model particles. Unbroken SUSY predicts the same masses for each particle and its superpartner, which is phenomenologically excluded, hence SUSY needs to be broken by a yet unknown mechanism at a high energy scale of $\sqrt{F}$. Superpartners receive masses of order m_S, where $m_S \ll \sqrt{F}$ and the phenomenologically relevant ranges are

$$1\,\mathrm{TeV} \lesssim \sqrt{F} \lesssim M_{\mathrm{Pl.}}, \quad 100\,\mathrm{GeV} \lesssim m_S \lesssim 10\,\mathrm{TeV}. \tag{1}$$

The gauge theory of local SUSY is called Supergravity and unifies gravity with the other fundamental interactions of the SM. In this scenario the spin 2 graviton also obtains a spin $\frac{3}{2}$ superpartner called the gravitino $\tilde{G}_\mu$. If SUSY is global then spontanous symmetry breaking would lead to a massles Goldstone-fermion or *Goldstino* Ψ. For local SUSY there is an anlogue of the Higgs-mechanism that turns the Goldstino Ψ into the longitudinal mode of the gravitino

$$\tilde{G}_\mu \to \tilde{G}_\mu + i\sqrt{4\pi}\frac{M_{\mathrm{Pl.}}}{F}\partial_\mu \Psi, \tag{2}$$

which in turn obtains as mass of

$$m_{\tilde{G}} = \sqrt{\frac{8\pi}{3}}\frac{F}{M_{\mathrm{Pl.}}}. \tag{3}$$

The gravitino is one possible dark matter candidate. Interactions (with matter) of the gravitino's transverse components are supressed by $M_{\mathrm{Pl.}}$ and only the longitudinal mode is relevant. In this exercise we restrict ourselves to effective gauge interactions of the schematic form

$$\mathcal{L}_{\text{int.}} = \frac{m_S}{F}\overline{\lambda}\gamma^\mu\gamma^\nu\Psi F_{\mu\,\nu}, \tag{4}$$

where $F_{\mu\,\nu}$ is the field-strength tensor of the $Z, W^\pm$-bosons and $\overline{\lambda}$ the fermionic superpartner of each gauge boson. We consider a scenario where all SM particles and their superpartners of common mass m_S couple with the same strength $g \approx 0.05$ to the gauge bosons.

a) Assume that at very high temperatures all SM particles and their superpartners were relativistic and present in the plasma. Using dimensional analysis estimate the cross section for $2 \leftrightarrow 2$ scattering processes involving a single graviton from equation (4). You may neglect all masses in the propagators. Find an expression for the freeze-out temperature T_f. *(2 points)*

b) In SUSY models there exists a discrete symmetry called R-parity. Under this symmetry all SM particles have parity $+1$ and all superpartners like the gravitino have -1. Assuming R-parity is conserved, how many SUSY particles other than the gravitino need to participate in the $2 \leftrightarrow 2$ reactions? Suppose your estimate for T_f was below the superpartner mass m_S, what would be the *true* freeze-out temperature in this case? *(2 points)*

c) Find g_* at the time of gravitino decoupling. Assume all SM particles and their superpartners were relativistic and in thermal equilibrium at T_f. With this information and your result from part (a) derive a bound on $\frac{m_S^2}{F}$ by demanding that the gravitino was relativistic at decoupling $\left(\frac{T_f}{m_{\tilde{G}}} \gg 1\right)$. Find the present day number density of relic gravitinos in terms of the photon number density. *(3 points)*
Hint: The gravitino has only two degrees of freedom as the interactions of the transverse modes are suppressed. To find g_* use the particle content of the MSSM, which has two Higgs doublets. For this estimate you can neglect the graviton.

d) Estimate the contribution $\Omega_{\tilde{G}}$ of non-relativistic relic gravitinos to the energy budget of the universe today. Use the present day photon number density $n_\gamma = 413\,\text{cm}^{-3}$ and $H_0 = h \times 100\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$. Derive a constraint on the gravitino mass by comparing $\Omega_{\tilde{G}}$ to the observed abundance of dark matter $\Omega_{DM}h^2 = 0.12$. *(2 points)*

While relic gravitinos produced via thermal freeze out could explain the measured relic abundance of dark matter, the resulting mass range is excluded by simulations of large scale structure formation, which requires dark matter masses $m_{DM} \gtrsim \mathcal{O}(1\,\text{keV})$.

H.8.3 Thermal average and scattering rates *(7 points)*

Until know we only used dimensional analysis to estimate the reaction rates of scattering processes. In this exercise we will derive a compact formula for the scattering rate, that only depends on the cross section and can be evaluated very fast by numerical means. The general definition for the scattering rate *density* of a process $a+b \to i+j$ reads

$$\Gamma_{a+b\to i+j} = \int \mathrm{d}\Pi_a \; \mathrm{d}\Pi_b \; \mathrm{d}\Pi_i \; \mathrm{d}\Pi_j \; f_a f_b \; (1 \pm f_i)(1 \pm f_j) \; \tilde{\delta}(\dots) \; \left|\mathcal{M}_{a+b\to i+j}\right|^2 , \tag{5}$$

where the f's indicate Fermi-Dirac or Bose-Einstein distributions and the plus (minus) sign in $1 \pm f$ corresponds to bosons (fermions). Furthermore the matrix element squared is $\left|\mathcal{M}_{a+b\to i+j}\right|^2$ and we introduce the following notation for the phase space volume element and the momentum conserving δ-distribution:

$$\mathrm{d}\Pi_x \equiv \frac{\mathrm{d}^3 p_x}{(2\pi)^3 2E_x}, \qquad \tilde{\delta}(\dots) \equiv (2\pi)^4 \delta^{(4)}\left(p_a + p_b - p_i - p_j\right) \tag{6}$$

The cross section is defined as

$$\sigma = \int \mathrm{d}\Pi_i \, \mathrm{d}\Pi_j \frac{\left|\mathcal{M}_{a+b\to i+j}\right|^2 \; \tilde{\delta}(\dots)}{4\sqrt{\left(p_a \cdot p_b\right)^2 - m_a^2 m_b^2}}. \tag{7}$$

a) First show that the two-body phase space can be written as

$$\int \mathrm{d}\Pi_a \, \mathrm{d}\Pi_b \; \tilde{\delta}(\dots) = \frac{1}{4\pi s}\sqrt{\left(p_a \cdot p_b\right)^2 - m_a^2 m_b^2}, \tag{8}$$

where $s = \left(p_a + p_b\right)^2$. *Hint: It is convenient to work in the center of mass frame and to note that the result is a Lorentz scalar.* *(2 points)*

b) Assume Maxwell-Boltzmann statistics for the distribution functions, which is valid for non-relativistic particles or a good approximation as long as $T \lesssim 3m$ and further neglect all chemical potentials. As long as there is no Fermi degeneracy or Bose condensation you may neglect the blocking factors $1 \pm f \approx 1$ too. Start by inserting a resolution of the identity

$$1 = \int \mathrm{d}^4 Q \; \delta^{(4)}\left(Q - p_a - p_b\right) \tag{9}$$

into equation (5) and make use of the cross section in (7) as well as of the following definitions for the Källén-triangle function λ and the modified Bessel function of the first kind K_1

$$\lambda(a,b,c) = (a-b-c)^2 - 4bc, \quad zK_1(z) = \int_0^\infty \mathrm{d}x \; \exp(-x)\sqrt{x^2 - z^2} \tag{10}$$

to derive

$$\Gamma_{a+b\to i+j} = \frac{T}{2(2\pi)^{10}} \int \mathrm{d}s \; s^{\frac{3}{2}} K_1\left(\frac{\sqrt{s}}{T}\right) \lambda\left(1, \frac{m_a^2}{s}, \frac{m_b^2}{s}\right) \sigma. \tag{11}$$

(5 points)

This scattering rate density scales as $\sim E^4$ instead of $\sim E$ like a physical rate should. The reason for this is that $\Gamma_{a+b\to i+j}$ appears in the following form of the Boltzmann equation $\frac{\mathrm{d}n_a}{\mathrm{d}t} + 3Hn_a = -\Gamma_{a+b\to i+j}\left(\frac{n_a^2}{n_a^{\mathrm{eq}.2}} - 1\right)$ implying that the thermal average of the cross section times relative velocity is $\langle\sigma|\vec{v}|\rangle = \frac{\Gamma_{a+b\to i+j}}{n_a^{\mathrm{eq}.2}}$.