Exercises General Relativity and Cosmology

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Hand in: Presence exercise

The Homework sheets have to be handed in on the first lecture of the week. There you will also find the new sheet.

The rooms and tutors for the exercise classes are as follows:

Monday, 14-16	Konferenzraum II, PI 1.049	Misha Gelenava
Tuesday, 8-10	Ü5, Raum 0.021, AVZ I	Leonardo Rydin
Wednesday, 14-16	Besprechungsraum, BCTP	Reza Safari
Thursday, 12-14	Seminarraum I, PI 2.005	Mahsa Barzegar

The tutorials start at monday, 24.4.2017.

If you have any questions feel free to contact me under fierro@th.physik.uni-bonn.de. You can find the addresses of the tutors on the course website:

`http://www.th.physik.uni-bonn.de/people/fierro/GRSS17/`

–Class exercises–

1 Minkowski diagrams

In the following we will work in 1+1 dimensional Minkowski spacetime and set $c = 1$.

a) Draw a spacetime diagram (x, t) and draw
   - a) an event.
   - b) a light-ray.
   - c) the worldline of an object that travels with constant velocity $v < 1$.
   - d) the worldline of an object that travels with constant velocity $v > 1$.
   - e) the worldline of an accelerated object.

b) Draw a spacetime diagram (x, t) of an observer $\mathcal{O}$ at rest. Into this spacetime diagram draw the worldline of an observer $\mathcal{O}'$ that travels with velocity $v < 1$ measured in the rest-frame of $\mathcal{O}$. Draw the coordinate axes of the spacetime diagram of $\mathcal{O}'$.
Hint: What is her time-axis? How do you then construct the space-axis?

c) You know that an object with length l' in the frame of the observer $\mathcal{O}'$ appears with length l to the observer $\mathcal{O}$ related to l' by

$$l = \sqrt{1 - v^2}\, l' \,. \tag{1}$$

In the following we consider the so-called garage paradox. We consider a car and a garage that have both length l at rest. The garage has a front-door (F) and a back-door (R). It is constructed in such a way, that it opens both doors when the front of the car arrives at the front door, closes both doors, if the back of the car reaches the front-door and opens both doors again, when the car leaves the garage (i.e. the front of the car arrives at the back-door). From the point of view of the garage the car is length-contracted and nicely fits into the garage. From the point of view of the car though, the garage is length-contracted and the car will not fit into it, but one expects that it will be destroyed by the doors. Resolve this paradox.
Hint: Draw a spacetime diagram in which the garage is at rest. What is the order in which the events appear for both observers?

d) Draw the spacetime diagram for an observer $\mathcal{O}$ sitting at the origin and draw her light cone. Mark the regions which are in space-like, time-like or light-like distance to her. How does the light cone change when you increase the speed of light? How does it look in the limit when $c \to \infty$?

2 The Lorentz group

We consider four-dimensional Minkowski space $\mathbb{R}^{1,3}$, which is $\mathbb{R}^4$ equipped with the Minkowski metric η. This is a symmetric, non-degenerate bilinear form $\eta : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ defined by

$$\eta(e_\mu, e_\nu) \equiv \eta_{\mu\nu} = \begin{cases} -1 & \text{for } \mu = \nu = 0 \\ +1 & \text{for } \mu = \nu = 1, 2, 3 \end{cases} \tag{2}$$

for the standard orthonormal basis $\{e_0, e_1, e_2, e_3\}$ on $\mathbb{R}^4$. Using linearity we then find

$$\eta(x, y) = x^{\mathrm{t}} \cdot \tilde{\eta} \cdot y \quad \text{for } x, y \in \mathbb{R}^{1,3}, \tag{3}$$

where $\tilde{\eta}$ is a matrix with entries $\eta_{\mu\nu}$. From now, we identify $\tilde{\eta}$ and η with each other and do not distinguish between them.
For $x, y \in \mathbb{R}^{1,3}$ we write $x \cdot y = \eta(x, y)$ and $x^2 = x \cdot x$. The postulates of special relativity imply that transformations Λ relating two inertial frames, so called Lorentz transformations, preserve the spacetime distance, i.e.

$$(x - y)^2 = (\Lambda(x - y))^2 \quad \text{for all} \quad x, y \in \mathbb{R}^{1,3} \,. \tag{4}$$

This leads to the definition of the *Lorentz group*

$$\mathrm{O}(1, 3) = \{\Lambda \in \mathrm{GL}(4, \mathbb{R}) \,|\, \Lambda^{\mathrm{t}} \eta \Lambda = \eta\} \,. \tag{5}$$

a) Show that $\Lambda \in \mathrm{O}(1, 3)$ indeed fulfills eq. (4).

b) Show that $\mathrm{O}(1,3)$ indeed is a group.

c) Embed the group of three-dimensional rotations into $\mathrm{O}(1,3)$.

Consider two inertial frames, K and K'. When K' moves in K with velocity v in positive x_1 direction, the Lorentz transformation from K to K' is $(c = 1, \gamma^{-1} = \sqrt{1 - v^2})$

$$\Lambda_{x_1}(v) = \begin{pmatrix} \gamma & -\gamma \cdot v & 0 & 0 \\ -\gamma \cdot v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Transformation of this type are called *boosts*. We introduce the *rapidity* ϕ by $v = \tanh \phi$.

d) Rewrite $\Lambda_{x_1}(v)$ from eq. (6) in terms of the rapidity.

e) Consider two succsessive boost, both in the x_1 direction but with different velocities. Find the rapidity of the composite boost. Deduce the relativistic rule for addition of velocities.