Physikalisches Institut
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Theoretische Physik

Exercise Sheet 1
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SS 17

Exercises General Relativity and Cosmology

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–Homework–

1 Linear algebra, indices and tensors (17.5 pts.)

In this exercise we want to review some basics of linear algebra, familiarize ourselves with the meaning of upper and lower indices and with the notion of tensors.
Consider a real vector space V of dimension $n < \infty$. For any vectorspace we define its dual vector space by $V^* = \{w : V \to \mathbb{R} \,|\, w \text{ linear}\}$. Upon introduction of a basis on V, $\mathbb{B}_1 = \{e_i \in V \,|\, \imath = 1, ..., n\}$, any vector $v \in V$ can be expanded in this basis. This means that there are unique numbers v^i, $i = 1, \ldots, n$ for which

$$v = \sum_{i=1}^{n} v^i e_i \equiv v^i e_i \tag{1}$$

holds. The numbers v^i are referred to as the components of v in the basis $\mathbb{B}_1$ and one sometimes identifies the list of numbers v^i with the vector v. This, however, works only as long as a specific basis is fixed, whereas the vector v itself exists independently of any basis. Once a basis of V has been chosen, there is a natural choice of basis of V^*. This is the dual basis $\mathbb{B}_1^* = \{e^i \in V^* \,|\, \imath = 1, ..., n\}$ defined by

$$e^i(e_j) = \delta^i_j \quad \forall i, j \in \{1, ..., n\}. \tag{2}$$

Now, any $w \in V^*$ can be written as

$$w = \sum_{i=1}^{n} w_i e^i \equiv w_i e^i, \tag{3}$$

in terms of unique numbers w_i, $i = 1, \ldots, n$.

a) Show that a dual vector is uniquely specified by its values on a basis of V, hence eq. (2) indeed specifies a set of dual vectors. Further show that $\mathbb{B}_1^*$ is indeed a basis of V^*, from which we deduce $V \simeq V^*$. *(2 points)*

b) Let $\mathbb{B}_2 = \{\tilde{e}_i \in V \,|\, \imath = 1, ..., n\}$ be a second basis of V and $\mathbb{B}_2^* = \{\tilde{e}^i \in V^* \,|\, \imath = 1, ..., n\}$ the associated dual basis. Write $e_i = (e_i)^j \tilde{e}_j$ and $e^i = (e^i)_j \tilde{e}^j$. Relate the components of $v \in V$ in $\mathbb{B}_1$ (and of $w \in V^*$ in $\mathbb{B}_1^*$) to those in $\mathbb{B}_2$ (and $\mathbb{B}_2^*$). *(1 point)*

c) With the same v and w, calculate $w(v)$ in both bases. Does the result depend on the basis? Deduce *(1 point)*

$$(e^i)_k (e_j)^k = \delta^i_j. \tag{4}$$

d) Let us now consider the bidual space $(V^*)^*$, which is the dual space of the dual space. Show that

$$\begin{aligned} \alpha : V &\longrightarrow (V^*)^* \\ v &\longmapsto \alpha(v) \text{ defined by } (\alpha(v))(w) = w(v) \text{ for all } w \in V^* \end{aligned} \tag{5}$$

is an isomorphism of vectorspaces. For surjectivity use that for linear maps the formula $\dim V = \dim(\ker \alpha) + \dim(\operatorname{im} \alpha)$ holds. *(2 points)*

Since α does not make reference to any basis, it is called a canonical isomorphism. Thus V and $(V^*)^*$ are regarded as the same space and are not distinguished.
Although we know $V \simeq V^*$ as well, there is in general no preferred choice of isomorphism between V and V^*. The situation changes if V is equipped with a symmetric non-degenerate[1] bilinear form $\beta : V \times V \longrightarrow \mathbb{R}$. Then it is natural to define the isomorphisms

$$\begin{aligned} \phi_1 : V &\longrightarrow V^* \\ v &\longmapsto \phi_1(v) = \beta(v, \cdot), \text{ declared by } (\phi_1(v))(w) = \beta(v, w) \text{ for all } w \in V, \\ \phi_2 : V^* &\longrightarrow V \\ w &\longmapsto \phi_2(w) \text{ defined by } \beta(\phi_2(w), v) = w(v) \text{ for all } v \in V. \end{aligned} \tag{6}$$

Write $\beta_{ij} = \beta(e_i, e_j)$ and define the numbers β^{ij} by $\beta^{ij}\beta_{jk} = \delta^i_j$.

e) Show that the components of $\phi_1(v)$ in $\mathbb{B}_1^*$ are related those in $\mathbb{B}_1$ by *(1 point)*

$$\phi_1(v)_i = \beta_{ij} v^j. \tag{7}$$

f) Show $\phi_2 \circ \phi_1 = \mathrm{id}_V$ and $\phi_1 \circ \phi_2 = \mathrm{id}_{V^*}$. Deduce that the components of $\phi_2(w)$ in $\mathbb{B}_1$ are related to those of w in $\mathbb{B}_1^*$ by *(2 points)*

$$\phi_2(w)^i = \beta^{ij} w_j. \tag{8}$$

This allows for changing back and forth between V and V^*. Application of ϕ_1 is called lowering an index and application of ϕ_2 raising an index.
So far we have encountered two objects, vectors (referred to as contravariant) and dual vectors (referred to as covariant). Let us now look at a generalization: A (k, l)-tensor T (k times contra- and l times covariant) is a multilinear map

$$T : \underbrace{V^* \times \ldots \times V^*}_{k \text{ times}} \times \underbrace{V \times \ldots \times V}_{l \text{ times}} \longrightarrow \mathbb{R}, \tag{9}$$

and the space of (k, l)-tensors is denoted $T^{k,l}$. The components of T with respect to the bases $\mathbb{B}_1$ and $\mathbb{B}_1^*$ are

$$T^{i_1 \ldots i_k}{}_{j_1 \ldots j_l} \equiv T(e^{i_1}, \ldots, e^{i_k}, e_{j_1}, \ldots, e_{j_l}). \tag{10}$$

The order of indices is significant, because T may answer differently on different arguments. Upper (lower) indices can be lowered (raised) with ϕ_1 (ϕ_2).

g) What type of tensors are scalars, dual vectors and β? What type of tensors are vectors and why is that so? *(1 point)*

[1] This is to ensure that ϕ_1 and ϕ_2 are isomorphisms.

h) Why is a tensor uniquely specified by its components? (*1 point*)

i) Find a basis of $T^{k,l}$ in terms of the basis vectors in $\mathbb{B}_1$ and $\mathbb{B}_1^*$. What is the dimension of $T^{k,l}$? (*2 points*)

j) Relate the components of T in $\mathbb{B}_1$ and $\mathbb{B}_1^*$ to those in $\mathbb{B}_2$ and $\mathbb{B}_2^*$. (*2 points*)

k) In special relativity spacetime $M = \mathbb{R}^{1,3}$ is equipped with the Minkowski metric η. What object in the previous exercises is η associated to? (*0.5 points*)

Since η is of Lorentzian signature, i.e. it has one negative and three positive eigenvalues, the indices are denoted by Greek and not Latin letters (which are reserved for Euclidean signature). In special relativity it is convenient to work with inertial frames. Consider two inertial frames — frame A with coordinates x^μ and frame B with y^μ — that are related by a Lorentz transformation $y^\mu = \Lambda^\mu{}_\nu\, x^\nu$.

l) Let the components of a (k,l)-tensor T in frame A be $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$. Show that the components in frame B are (*2 points*)

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \ldots \nu_l'} = \Lambda^{\mu_1'}{}_{\mu_1} \ldots \Lambda^{\mu_k'}{}_{\mu_k} (\Lambda^{-1})^{\nu_1}{}_{\nu_1'} \cdots (\Lambda^{-1})^{\nu_l}{}_{\nu_l'} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}. \tag{11}$$

2 Classical electrodynamics (22 pts.)

In this exercise we consider the field theoretical formulation of classical electrodynamics. The electromagnetic field is described in terms of a vector field

$$A : \mathbb{R}^{1,3} \longrightarrow \mathbb{R}^{1,3}, \tag{12}$$

with components A^μ. These are related to the components of the field strength tensor F via

$$F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu. \tag{13}$$

The particles of mass m_i and charge q_i are described by their trajectories,

$$\begin{aligned} x_i : I_k \subset \mathbb{R} &\longrightarrow \mathbb{R}^{1,3} \quad \text{for } i = 1, ..., N = \text{number of particles} \\ \sigma_i &\longmapsto x_i(\sigma_i), \end{aligned} \tag{14}$$

which are parameterised by an arbitrary curve parameter σ_i and whose components are x_i^μ. The action is

$$\begin{aligned} S[x_i, A] = &- \sum_{i=1}^{N} \int_{I_k} \mathrm{d}\sigma_i \left(m_i \sqrt{-\eta_{\alpha\beta} \dot{x_i}^\alpha(\sigma_i) \dot{x_i}^\beta(\sigma_i)} - q_i\, A_\alpha(x_i(\sigma_i)) \dot{x_i}^\alpha \right) \\ &- \frac{1}{4} \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\, F_{\alpha\beta}(x) F^{\alpha\beta}(x), \end{aligned} \tag{15}$$

where $\dot{x_i} = \frac{\mathrm{d}}{\mathrm{d}\sigma_i} x_i$ and integration on Minkowski space is the same as on $\mathbb{R}^4$.

a) Shortly comment on the significance of each term in S. (*1 point*)

b) Take the variation of S with respect to x_i^μ in order to derive the Einstein-Lorentz equation *(2 points)*

$$m_i \frac{\mathrm{d}}{\mathrm{d}\sigma_i} \frac{\dot{x}_i{}^\mu(\sigma_i)}{\sqrt{-\eta_{\alpha\beta}\dot{x}_i{}^\alpha(\sigma_i)\dot{x}_i{}^\beta(\sigma_i)}} = q_i\, F^\mu{}_\nu(x_i(\sigma_i))\dot{x}_i{}^\nu. \tag{16}$$

c) Rewrite the second term in S, the term where q_i appears, in terms of the charge-current density *(1 point)*

$$j^\mu(x) = \sum_{i=1}^N q_i \int \mathrm{d}\sigma_i\, \delta^{(4)}(x - x_i(\sigma_i))\, \dot{x}_i{}^\mu(\sigma_i). \tag{17}$$

d) Take the variation of S with respect to A_μ to derive the inhomogenous Maxwell's equations *(2 points)*

$$\frac{\partial}{\partial x^\mu} F^{\nu\mu}(x) = j^\nu(x). \tag{18}$$

e) Use the definition of the field strength tensor, eq. (13), to show the homogenous Maxwell's equations *(2 points)*

$$\frac{\partial}{\partial x^\alpha} F_{\mu\nu} + \frac{\partial}{\partial x^\mu} F_{\nu\alpha} + \frac{\partial}{\partial x^\nu} F_{\alpha\mu} = 0. \tag{19}$$

Now take $A^\mu = (\phi, \vec{A})$ with ϕ and $\vec{A}$ such that $\vec{B} = \operatorname{rot}\vec{A}$ and $\vec{E} = -\operatorname{grad}\phi - \dot{\vec{A}}$. Further, $j^\mu = (\rho, \vec{j})$ with the charge-density ρ and current-density $\vec{j}$.

f) Express the components of the field strength tensor, $F_{\alpha\beta}$, in terms of the components of the electric and magnetic field, $\vec{E}$ and $\vec{B}$. *(1 point)*

g) Show that eq. (18) indeed gives the inhomogenous Maxwell's equations: *(2 points)*

$$\operatorname{div}\vec{E} = \rho, \qquad \operatorname{rot}\vec{B} - \dot{\vec{E}} = \vec{j}. \tag{20}$$

h) Show that eq. (19) indeed gives the homogenous Maxwell's equations: *(2 points)*

$$\operatorname{div}\vec{B} = 0, \qquad \operatorname{rot}\vec{E} + \dot{\vec{B}} = 0. \tag{21}$$

i) Parameterise eq. (16) by time, i.e. $\sigma = x^0 = t$, and show that it reduces to *(2 points)*

$$m \frac{\mathrm{d}}{\mathrm{d}t} \frac{\vec{v}}{\sqrt{1 - v^2}} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{22}$$

Now we consider the canonical energy-momentum tensor of a field $\phi(x)$ given by

$$T_\mu^\nu = \delta_\mu^\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi \tag{23}$$

For the next exercises, we consider the Maxwell electromagnetic contribution $\mathcal{L}_{EM}$ of (15) with $j^\mu = 0$.

j) Compute the canonical-energy momentum tensor for $\mathcal{L}_{EM}$. *(1 point)*

In the previous item you will notice that such a tensor is not symmetric. To repair this add to $T^{\mu\nu}_{EM}$ an extra term as follows

$$\widehat{T}^{\mu\nu}_{EM} = T^{\mu\nu}_{EM} + \partial_\lambda F^{\mu\lambda} A^\nu. \tag{24}$$

k) Use the appropriate equations of motions to verify that $\partial_\mu T^{\mu\nu}_{EM} = 0$. Why is it $\widehat{T}_{EM}$ an equally good energy-momentum tensor? *(2 points)*

l) Express the components $\widehat{T}^{00}_{EM}$ and $\widehat{T}^{0i}_{EM}$ in terms of the 3-vector fields $\vec{E}$ and $\vec{B}$. *(2 points)*

Since charged particles and the electromagnetic field interact, these can exchange energy and momentum. Therefore we would expect $\partial_\mu \widehat{T}^{\mu\nu}_{EM} \neq 0$ for the case $j^\mu \neq 0$. To make the energy-momentum conservation law hold, we need to add the energy-momentum tensor contribution of the particles. For an a-th particle this is given by

$$T^{(a)}_{\mu\nu} = \frac{p^{(a)}_\mu p^{(a)}_\nu}{p^{0(a)}} \delta^{(3)}(\vec{x} - \vec{x}^{(a)}(\sigma_a)). \tag{25}$$

Consider the following total energy momentum tensor

$$T^{\mu\nu} = \sum_{a=1}^{N} T^{(a)\mu\nu} + \widehat{T}^{\mu\nu}_{EM}. \tag{26}$$

m) Show that for the particle-field interacting case, $\partial_\mu T^{\mu\nu} = 0$ is satisfied. *(2 points)*