Physikalisches Institut
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Theoretische Physik

Exercise Sheet 2
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SS 17

Exercises General Relativity and Cosmology

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–Homework–

1 Matter content of the universe as a perfect fluid (10 points)

For a system of N discrete point particles the energy-momentum tensor takes the form

$$T_{\mu\nu} = \sum_{a=1}^{N} \frac{p_\mu^{(a)} p_\nu^{(a)}}{p^{0(a)}} \delta^{(3)}(\vec{x} - \vec{x^{(a)}}(\sigma_a)), \tag{1}$$

where the index a labels an a-th particle.

a) Show that, for a dense collection of particles with isotropically distributed velocities, we can smooth over the individual particle worldlines to obtain (*2 points*)

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2}$$

Hint: What would be the definition of p and ρ in this case?.

Note that $T^{\mu\nu}$ an be understood as the stress-energy tensor of a perfect fluid with mass density ρ and pressure p.

b) Show that $T^{\mu\nu}$ can also be written as (*2 points*)

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}, \tag{3}$$

where U^μ can be tought as the components of the four-velocity of the fluid.

c) From the nonrelativistic limit of the conservation of the energy-momentum tensor, $\partial_\mu T^{\mu\nu} = 0$, deduce the Euler's equations (*3 points*)

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0, \tag{4}$$

$$\rho\Big[\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v}\Big] = -\nabla p. \tag{5}$$

Hint: The nonrelativistic limit is given by $U^\mu = (1, v^i)$, $|v^i| \ll 1, p \ll \rho$. Project the equation into pieces along and orthogonal to the four-velocity by contraction with U_ν and $P^\rho_\nu = \delta^\rho_\nu + U^\rho U_\nu$.

d) For the nonrelativistic limit argue that the 'fluid' follows the ideal gas law (*2 points*).

e) Show that in the highly relativistic limit, the trace T^μ_μ gives the equation of state (*1 point*).

$$p = \frac{1}{3}\rho \tag{6}$$

2 Angular momentum and Noether currents (8 points)

Consider a classical field theory of a scalar field $\phi(x)$ defined over the Minkowski space $\mathbb{R}^{1,3}$ with an action given by

$$S = \int_{\mathbb{R}^{1,3}} \mathrm{d}^4 x \mathcal{L}(\phi, \partial_\mu \phi), \tag{7}$$

which is invariant under Lorentz transformations $x^\mu \mapsto \Lambda^\mu_\nu x^\nu$.

a) Show that the parameter of an infinitesimal transformation *(1 point)*

$$x^\mu \mapsto (\delta^\mu_\nu + \omega^\mu_\nu) x^\nu, \tag{8}$$

has to be antisymmetric in its upper indices, i.e. $\omega^{\mu\nu} + \omega^{\nu\mu} = 0$.

b) Show, that the conserved currents of such an infinitesimal transformation are given by *(5 points)*

$$\mathcal{M}^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}, \tag{9}$$

where $T^{\mu\nu}$ is the canonical energy momentum tensor.

c) What are the conserved charges? Show that *(2 points)*

$$\frac{d}{dt} \int_{\mathbb{R}^3} \mathrm{d}^3 x x^i T^{00} = \text{const.} \tag{10}$$

3 Tensor gymnastics (12 points)

Let A be a real valued $n \times n$ dimensional matrix. For the later content of the lecture, the following identity will be useful

$$\det A = \exp\Big(\operatorname{tr} \log A\Big). \tag{11}$$

a) Prove (11). *(6 points)*

In special relativity, a pseudotensor is an object that transform under Lorentz transformations, $x^{\mu'} \mapsto \Lambda^{\mu'}_\mu x^\mu$, just like a tensor up to a determinant factor.

$$\widetilde{T}^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = (\det \Lambda) \Lambda^{\mu'_1}{}_{\mu_1} \cdots \Lambda^{\mu'_k}{}_{\mu_k} \Lambda^{\nu_1}{}_{\nu'_1} \cdots \Lambda^{\nu_l}{}_{\nu'_l} \widetilde{T}^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}. \tag{12}$$

In particular, the Levi-Civita symbol $\widetilde{\epsilon}_{\mu_1 \cdots \mu_n}$ is an example of pseudotensors, where

$$\widetilde{\epsilon}_{\mu_1 \cdots \mu_n} = \begin{cases} +1 \text{ if } \mu_1 \cdots \mu_n \text{ is an even permutation} \\ -1 \text{ if } \mu_1 \cdots \mu_n \text{ is an odd permutation} \\ 0 \text{ otherwise} \end{cases} \tag{13}$$

b) Show that under a coordinate transformation in special relativity, $\widetilde{\epsilon}_{\mu'_1 \cdots \mu'_n} = \widetilde{\epsilon}_{\mu_1 \cdots \mu_n}$. *(2 points)*

Consider adding to the Lagrangian for electromagnetism of Homework 1 an additional term of the form

$$\widetilde{\mathcal{L}}_{EM} = \widetilde{\epsilon}_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}. \tag{14}$$

c) Express $\widetilde{\mathcal{L}}_{EM}$ in terms of the 3-vector fields $\vec{E}$ and $\vec{B}$. *(2 points)*

d) Show that including $\widetilde{\mathcal{L}}_{EM}$ does not affect Maxwell's equations. Can you think of a deep reason for this? *(2 points)*