Physikalisches Institut
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Theoretische Physik

Exercise Sheet 3
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SS 17

Exercises General Relativity and Cosmology

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–Homework–

1 Elements of Topology (15 points)

In this exercise, we will briefly cover the definitions and essential notions from topology, as this will be a key ingredient for the definition of a manifold. Let us recall the definition: A topological space consists of a set X and a set $\mathcal{T}$ of subsets of X satisfying the conditions:

- The empty subset of X is an element of $\mathcal{T}$.
- $X \in \mathcal{T}$.
- If $\{U_\alpha\}$ is a set of elements of $\mathcal{T}$, then $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- If $U_\alpha \in \mathcal{T}$ and $U_\beta \in \mathcal{T}$, then $U_\alpha \cap U_\beta \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called the open subsets of X, and $\mathcal{T}$ is called a topology on X. An open set containing $x \in X$ is called a neighborhood of x. A closed subset of X is the complement of an open subset.

a) Let $X = \{1, \ldots, n\}$ and $\mathcal{T}$ be the set of all the possible subsets of X. Show that $\mathcal{T}$ is a topology. *(1 point)*

b) $\mathbb{R}^n$ has a natural topology. Let $a \in \mathbb{R}^n$ and $r > 0$. Let $B(a;r)$ denote the open ball in $\mathbb{R}^n$ of a radius r centered at a: *(2 points)*

$$B(a;r) = \{x \in \mathbb{R}^n : |x - a| < r\}$$

Let $\mathcal{T}$ be the set of all arbitrary unions of balls $B(a;r)$. Show that $\mathcal{T}$ is a topology on $\mathbb{R}^n$. This topology is referred to the Euclidean topology.

c) Let X be a topological space, and let $Y \subset X$ be a subset. Then Y inherits a topology from X called the subspace topology, and we refer to Y with this topology as a subspace of X. The open subsets $\mathcal{T}_Y$ of Y are defined to be the intersections of the open subsets of X with Y : *(2 points)*

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}_X\}.$$

Show that $\mathcal{T}_Y$ is a topology.

d) Let $X = \mathbb{R}^2$ with $\mathcal{T}_X =$ the euclidean topology, and $Y = S^1$ with the subset topology $\mathcal{T}_Y$ induced by $\mathcal{T}_X$. What would be the open sets of $\mathcal{T}_Y$? *(1 point)*

e) Let $X = \mathbb{R}$ with the euclidean topology. Let Y be (separate cases): (*1 point*)

$$(a, b), \quad [a, b), \quad (a, b], \quad [a, b]$$

What would be the subset topology $\mathcal{T}_Y$ (for each case) induced by the topology of X?

Topology also gives a way to formalize the notion of continuity. Let X and Y be topological spaces. A mapping $f : X \to Y$ is *continuous* if for all open subsets $V \subset Y$, its inverse image $f^{-1}(V)$ is open in X.

f) A function $f : \mathbb{R} \to \mathbb{R}$ is continuous in the sense of topology if and only if it is continuous in the sense of calculus; that is, given any a$\in \mathbb{R}$ and any $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Prove such a statement. (*4 points*)

g) Suppose X, Y, Z are topological spaces and $f : X \to Y, g : Y \to Z$ are continuous. Prove that $g \circ f : X \to Z$ is continuous. (*2 points*)

Finally, an isomorphism in the category of topological spaces is known as an *homeomorphism*[1], more precisely: A continuous map $f : X \to Y$ between topological spaces is a homeomorphism if it has a continuous inverse $g : Y \to X$. The spaces X and Y are homeomorphic if there exists an homeomorphism $f : X \to Y$.

h) Show that a circle $S^1_a = \{(x, y) =: x^2 + y^2 = r_a^2\}$ is homeomorphic to a circle $S^1_b = \{(x, y) =: x^2 + y^2 = r_b^2\}$. (*1 point*)

i) Show that the unit circle $S^1 = \{(x, y) =: x^2 + y^2 = 1\}$ is homeomorphic to the square $\square = \{(x, y) : x = \pm 1, 1 \leq y \leq 1, \text{ or } -1 \leq x \leq 1, y = \pm 1\}$. (*1 point*)

2 Getting used to Manifolds (15 points)

In General Relativity, the concept of manifolds is of central importance for the description of spacetime. This exercise is devoted to build up some intuition for these objects. To this end, let us first recall the definition: A differentiable manifold of dimension n is a topological space M, such that the space can be covered with a set of open sets $\{U_\alpha\}$ and for every α we have a homeomorphism $h_\alpha : U_\alpha \longrightarrow V_\alpha$ to an open set $V_\alpha \subset \mathbb{R}^n$. A pair (U_α, h_α) is called a chart and a set of charts covering M is called an atlas. Further, for every pair (α, β) such that $U_\alpha \cap U_\beta \neq \emptyset$ the transition function $h_{\alpha\beta} \equiv h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \longrightarrow h_\alpha(U_\alpha \cap U_\beta)$ is required to be a diffeomorphism.

a) Consider $\mathbb{R}^n$ itself as a topological space and explicitly construct an atlas to show that it is a manifold. (*2 points*)

   Hint: One chart is enough.

b) Consider the unit circle $S^1 \subset \mathbb{R}^2$ and show it is a manifold. For this pick the north and south poles $(0, 1)$ and $(0, -1)$ in S^1 and the maps $\phi_1 : S^1 - (0, 1) \to \mathbb{R}$, $\phi_2 : S^1 - (0, -1) \to \mathbb{R}$ given by: (*2 points*)

$$\phi_1(x, y) = \frac{x}{1 - y}, \quad \phi_2(x, y) = \frac{x}{1 + y}.$$

[1]Do not to confuse this with an homomorphism.

The result in a) not suprising since manifolds are constructed such that they locally look like the $\mathbb{R}^n$ and of course does $\mathbb{R}^n$ locally look like $\mathbb{R}^n$.
A manifold is called topologically trivial if it can be continuously shrunk to a point, an example is $\mathbb{R}^n$. One may think that the property of $\mathbb{R}^n$ being topologically trivial allowed for covering it with one chart only. This is not true, as we will see in the next item.

c) Consider the infinitely long cyclinder M given by its embedding in $\mathbb{R}^3$,

$$M = \{(R\cos\phi, R\sin\phi, t) \,|\, \phi \in [0, 2\pi),\ t \in (-\infty, \infty),\ R > 0\}\ . \tag{1}$$

Although M is topologically non-trivial — it can be shrunk to a circle but not to a point — it can be covered with a single chart only. Construct such a chart explitly. (*4 points*)

Hint: It may be easier to first think about how M can be mapped to the punctured complex plane, $\mathbb{C} \setminus \{0\}$.

d) Now consider the two-dimensional torus given by its embedding in $\mathbb{R}^3$,

$$T^2 = \{((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta) \,|\, \theta, \phi \in [0, 2\pi),\ R > r > 0\}\ . \tag{2}$$

Explicitly construct an atlas to show that T^2 is a manifold. (*4 points*)

e) The n-dimensional real projective space $\mathbb{RP}^n$ is the set of all ordered $(n+1)$-tuples of complex numbers: (*3 points*)

$$\{x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x \neq (0, \ldots, 0)\},$$

where we identify pairs which are scalar multiples of each other, i.e. $x \sim \lambda x'$ for some $\lambda \in \mathbb{R}^* = \mathbb{R} - 0$. Show that $\mathbb{RP}^n$ is a manifold. For this consider the U_i cover with $x_i \neq 0$ for $x \in U_i$, for a given $i \in \{1, \ldots, n\}$. Use the maps

$$\phi_i : U_i \to \mathbb{R}^n, \quad (x_0, \ldots, x_n) \mapsto \left(\frac{x_0}{x_i}, \cdots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}\right).$$