Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 4
15.5.2016
SS 17

Exercises General Relativity and Cosmology

Priv.-Doz. Stefan Förste, Cesar Fierro

Hand in: 22.5.2017

Just a reminder: The solutions must be handed in in groups of two to three students.

–Homework–

1 Tangent & cotangent vectors in General Relativity (10 points)

In this part we make a comeback to part 1 of the exercise sheet 1. Now we make the application for general relativity, where the spacetime is described by a differentiable manifold M equipped with a *Lorentz metric*[1] g.

a) Why can spacetime not play the role of V in general relativity? (*2 points*)

Instead, the tangent space T_pM at the point $p \in M$ plays the role of V (pointwise). Given coordinates on a patch $U \subset M$ around p, $x^\mu : U \longrightarrow \mathbb{R}^4$ with $\mu = 1, \ldots, 4$, the tangent space is spanned by the partial derivates with respect to the coordinates, i.e. $\partial_\mu = \frac{\partial}{\partial x^\mu}$. The duals of ∂_μ are denoted by $\mathrm{d}x^\mu$, they are elements of the cotangent space T_p^*M.

b) Does μ in x^μ label a vector or dual vector component, or a set of maps? (*0.5 points*)

c) What object in H 1.1 is ∂_μ associated to? (*1 point*)

d) Express $\partial'_\mu = \frac{\partial}{\partial y^\mu}$ in terms of the ∂_μ. (*2 points*)

e) In item b) of H 1.1 we have looked at a change of basis. What is the analog to this here: The change from x^μ to y^μ or from ∂_μ to ∂'_μ? (*1 point*)

f) Express $\mathrm{d}y^\mu$ (the dual of ∂'_μ) in terms of the $\mathrm{d}x^\mu$. (*1 point*)

We have stated above, that T_pM plays the role of V. This means that V now depends on the point in spacetime, thus it is natural to consider tensor fields,

$$\mathcal{T}^{k,l} : M \longrightarrow T^{k,l}. \tag{1}$$

A tensor field of (k,l) type assigns a (k,l)-tensor to each point in spacetime.

g) Let the components of a (k,l)-tensor T associated to the basis $\{\partial_\mu\}$ of T_pM be $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$. What would be the components of T in a basis $\{\partial_{\mu'}\}$? (*1 point*)

In particular, at a given $p \in M$ the Lorentz metric $g_p : T_pM \times T_pM \to \mathbb{R}$ is a type (0,2) which satisfies the following axioms at each point $p \in M$:

[1] Its definition is given below.

- $g_p(U,V) = g_p(V,U)$,
- if $g_p(U,V) = 0$ for any $U \in T_pM$, then $V = 0$,
- one of the eigenvalues of the matrix form of $(g_{\mu\nu}(p))$ in $g_p = g_{\mu\nu}(p)\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ is negative, the rest are positive.
- The map $p \mapsto g_p$ is smooth.

h) According to H1.1, what object is associated to g_p? What is $g_{\mu\nu}(p)$ in this case? (*1 point*)

i) What would be the physical meaning associated to the third axiom for a Lorentz metric? (*0.5 points*)

If g is a Lorentz metric, (M,g) is called a Lorentz manifold. For such a case, the elements of T_pM are divided into three classes:

- $g_p(U,U) > 0 \Rightarrow U$ is spacelike,
- $g_p(U,U) = 0 \Rightarrow U$ is lightlike,
- $g_p(U,U) < 0 \Rightarrow U$ is timelike.

2 Pullbacks & Pushforwards (13 points)

In local coordinates $\{y^\alpha\}$ on N the metric tensor can be expanded as

$$g = g_{\alpha\beta}\,\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta \tag{2}$$

in terms of smooth functions $g_{\alpha\beta}$. Examples are $\mathbb{R}^n$ equipped with the standard euclidean metric, which is just what we call $\mathbb{R}^n$, or $\mathbb{R}^n$ equipped with the Minkowski metric, which is n-dimensional Minkowski space $\mathbb{R}^{1,n-1}$. This also shows that one and the same manifold can be equipped with different metrics, by which it is made into different objects. Consider a manifold N equipped with a metric g and a second manifold M with local coordinates $\{x^\mu\}$. If we further have a smooth map $\varphi : M \longrightarrow N$ we can use this map induce a metric for M. For that we need to introduce a couple of new concepts:

The tangent bundle TM of a manifold M assembles all tangent vectors in M, it is defined as

$$TM = \bigcup_{p\in M} \{p\} \times T_pM = \{(p,v) : p \in M, v \in T_pM\}. \tag{3}$$

The map φ also induces a natural map $\varphi_* : TM \to TN$ called the *pushforward* of φ, which makes the following diagram commute

$$\begin{array}{ccc} TM & \xrightarrow{\varphi_*} & TN \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\varphi} & N \end{array} \quad , \qquad \text{i.e. } \pi \circ \varphi_* = \varphi \circ \pi.$$

Here $\pi(p,v)=p$. Let $V \in T_pM$. Then the action of φ_* on V will by given by

$$\varphi_* V = \frac{\partial y^\alpha}{\partial x^\mu} V^\mu \frac{\partial}{\partial y^\alpha}\Big|_{\varphi(p)} . \tag{4}$$

Another important concept is that of the *pullback*, for a tensor field $A \in \mathcal{T}^{0,l}$ the pullback $\varphi^* A$ is given by

$$\varphi^* A(p)(V_1, \ldots, V_l) = A(\varphi(p))(\varphi_* V_1, \ldots, \varphi_* V_l). \tag{5}$$

This gives the so called *induced metric* on M, which is denoted as $\varphi^* g$. With g as in eq. (2), the induced metric locally reads

$$\varphi^* g = \left[g_{\alpha\beta} \left(\frac{\partial y^\alpha}{\partial x^\mu} \right) \left(\frac{\partial y^\beta}{\partial x^\nu} \right) \right] \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \ . \tag{6}$$

a) Use the pushforward φ_* action in (4) to derive (6). *(1 point)*

Consider the two-sphere S^2 embedded in $\mathbb{R}^3$,

$$S^2 = \{ R(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \,|\, \phi \in [0, 2\pi),\ \theta \in [0, \pi),\ R > 0 \}. \tag{7}$$

b) How do you need to restrict the domain of θ and ϕ so that the spherical coordinates

$$\varphi_s^{-1}(\phi, \theta) : \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} , \tag{8}$$

provide a chart on the sphere? *(1 point)*

Let ι be the inclusion map

$$\begin{aligned} \iota : \qquad S^2 &\longrightarrow \mathbb{R}^3 \\ (\theta, \phi) &\longmapsto R(\cos\phi \sin\theta, \sin\phi\sin\theta, \cos\theta) \end{aligned} \tag{9}$$

c) Consider the following points $\{(0,0), (\pi, 0), (0, \pi/2)\} \in S^2$. Let $X_\theta = \partial_\theta$, $X_\phi = \partial_\phi$. What would be the pushforward $\iota_* X_\theta(p)$ and $\iota_* X_\phi(p)$ for each of these points? *(1 point)*

d) Calculate the induced metric on S^2. *(3 points)*

Now consider the two-dimensional torus given by its embedding in $\mathbb{R}^3$,

$$T^2 = \{ ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta) \,|\, \theta, \phi \in [0, 2\pi),\ R > r > 0 \} \ . \tag{10}$$

e) The embedding of T^2 in $\mathbb{R}^3$ is given in eq. (10). Use the corresponding inclusion map to calculate the induced metric on T^2. *(3 points)*

f) In cosmology the so called *de Sitter space* will be of importance. This space is cut out of five-dimensional Minkowski space $\mathbb{R}^{1,4}$ — with coordinates u, w, x, y, z, with u being timelike — by the hyperboloid equation

$$-u^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2, \quad \alpha \in \mathbb{R} \ . \tag{11}$$

On de Sitter space we introduce coordinates t, χ, θ, ϕ and embed it in $\mathbb{R}^{1,4}$ by

$$\begin{aligned} &u = \alpha\sinh(t/\alpha), \quad w = \alpha\cosh(t/\alpha)\cos\chi, \quad x = \alpha\cosh(t/\alpha)\sin\chi\cos\theta \\ &y = \alpha\cosh(t/\alpha)\sin\chi\sin\theta\cos\phi, \quad z = \alpha\cosh(t/\alpha)\sin\chi\sin\theta\sin\phi \ . \end{aligned} \tag{12}$$

Calculate the induced metric on de Sitter space. *(4 points)*

3 Vector fields & tensor acrobatics (15 pts.)

A smooth vector field X on a manifold M fulfills the two conditions

$$\begin{array}{lll} \text{Linearity:} & X(\alpha f+\beta g)=\alpha X(f)+\beta X(g) & \text{with} \quad \alpha,\beta\in\mathbb{R},\ f,g\in C^\infty(M) \\ \text{Leibniz rule:} & X(f\cdot g)=f\cdot X(g)+g\cdot X(f) & \text{with} \quad f,g\in C^\infty(M)\ . \end{array} \tag{13}$$

In general, maps with the properties (13) are called *derivations*. Given two vector fields X and Y we define a new vector field $[X,Y]$, the *Lie bracket* or *commutator* of X and Y, by

$$[X,Y](f)=X(Y(f))-Y(X(f)) \quad \text{for} \quad f\in C^\infty(M). \tag{14}$$

a) Show in two ways that $[X,Y]$ is indeed a vector field:
   i) Prove that $[X,Y]$ is a derivation. *(3 points)*
   ii) Write $[X,Y]$ in terms of components and show that they transform as those of a vector field unter change of coordinates. *(2 points)*

   Note that neither XY nor YX is a vector field.

b) Show that the Lie bracket is
   i) skew-symmetric, $[X,Y]=-[Y,X]$, and *(1 point)*
   ii) satisfies the Jacobi identity, $[[X,Y],Z]+[[Z,X],Y]+[[Y,Z],X]=0$. *(2 points)*

c) Consider $\mathbb{R}^2$ equipped with some coordinates x^1,x^2. Calculate the Lie bracket of the coordinate vector fields $\partial_1=\frac{\partial}{\partial x^1}$ and $\partial_2=\frac{\partial}{\partial x^2}$. *(1 point)*

d) Find an example of two nowhere-vanishing, (at each point) linearly independent vector fields in $\mathbb{R}^2$ whose Lie bracket does not vanish. Note that these two vector fields provide a basis for the tangent space at each point. Due to your findings in item c) they can, however, not be coordinate vector fields. *(3 points)*

Let $T\in T^{0,k}(V)$ in a given vector space V. We denote $\mathrm{Sym}(T)$ as the symmetrized form of T, where its components are given by

$$\mathrm{Sym}(T)_{\mu_1,\dots\mu_k}=T_{(\mu_1,\dots,\mu_k)}=\frac{1}{k!}\sum_{\sigma\in S_k}T_{\mu_{\sigma(1)},\dots\mu_{\sigma(k)}}. \tag{15}$$

Symilarly, we define the fully antisymmetrized form of T as the tensor $\mathrm{Alt}(T)$ such that

$$\mathrm{Alt}(T)_{\mu_1,\dots\mu_k}=T_{[\mu_1,\dots,\mu_k]}=\frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}\sigma\cdot T_{\mu_{\sigma(1)},\dots\mu_{\sigma(k)}}. \tag{16}$$

We can make an analogous definition for tensors in $T^{l,0}(V)$ by raising all of the indices in (15) and (16).

1. For a $T\in T^{0,2}(V)$, show that $T=\mathrm{Sym}(T)+\mathrm{Alt}(T)$. *(1 point)*
2. Why is that for $T\in T^{0,3}(V)$, $\quad T\neq\mathrm{Sym}(T)+\mathrm{Alt}(T)$? *(0.5 points)*
3. Prove for $X\in T^{2,0}(V)$ and $Y\in T^{0,2}(V)$ the following: *(1.5 points)*
   - $X^{(\mu\nu)}Y_{\mu\nu}=X^{(\mu\nu)}Y_{(\mu\nu)}$,
   - $X^{[\mu\nu]}Y_{\mu\nu}=X^{[\mu\nu]}Y_{[\mu\nu]}$,
   - $X^{[\mu\nu]}Y_{(\mu\nu)}=0$.