Physikalisches Institut
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Theoretische Physik

Exercise Sheet 5
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SS 17

Exercises General Relativity and Cosmology

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Special announcement: Due to *Christi Himmelfahrt*, the Group 4 tutorial will be shifted to Friday 26.05. (14:00-16:00) and will take place in the BCTP *Besprechungsraum.*

–Homework–

1 Differential forms (20 points)

A *differential form* of order k or an k-form is a totally antisymmetric tensor of type $(0, k)$. Let us define the *wedge product* $\wedge$ of k one-fomrs by the totally antisymmetric tensor product

$$\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_k} = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \mathrm{d}x^{\mu_{\sigma(1)}} \otimes \ldots \otimes \mathrm{d}x^{\mu_{\sigma(k)}}. \tag{1}$$

a) Verify that the wedge product satisfies the following (*1 point*)

- $\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_k} = 0$ if some index μ appears at least twice.
- $\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_k} = \mathrm{sgn}(\sigma)\, \mathrm{d}x^{\mu_{\sigma(1)}} \wedge \ldots \wedge \mathrm{d}x^{\mu_{\sigma(k)}}$.
- $\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_k}$ is linear in each $\mathrm{d}x^{\mu}$.

If we denote a vector space of k-forms at $p \in M$ by $\Omega_p^k(M)$, the set of k-forms (1) is a basis of $\Omega_p^k(M)$ and an element $\omega \in \Omega_p^k(M)$ is expanded as

$$\omega = \frac{1}{k!} \omega_{\mu_1 \ldots \mu_k}\, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_k}, \tag{2}$$

where $\omega_{\mu_1 \cdots \mu_k}$ are taken totally antisymmetric, reflecting the antisymmetry of the basis. Since there are $\binom{n}{k}$ choices of the set $(\mu_1, \ldots, \mu_k)$ out of $(1, \ldots, n)$ in (1), for an n-dimensional manifold M, then $\dim \Omega_p^k(M) = \binom{n}{k}$. We define the exterior product of a k-form and a l-form $\wedge : \Omega_p^k(M) \times \Omega_p^l(M) \to \Omega_p^{k+l}(M)$ by a trivial extension. Let $\omega \in \Omega_p^k(M)$ and $\xi \in \Omega_p^l(M)$, the action of the $(k+l)$-form on $k+l$ vectors is defined by

$$\omega \wedge \xi = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\omega \otimes \xi). \tag{3}$$

If $k + l > n, \omega \wedge \xi = 0$. With the exterior product we can define an algebra on the following vector space

$$\Omega_p^*(M) = \bigoplus_{k=0}^{n} \Omega_p^k(M). \tag{4}$$

We refer to (4) as the spaces of all diferential forms at $p \in M$.

b) Let $\omega \in \Omega^k_p(M), \xi \in \Omega^l_p(M)$ and $\eta \in \Omega^r_p(M)$. Show that (*2 points*)

- $\omega \wedge \omega = 0$, if k is odd,
- $\omega \wedge \xi = (-1)^{kl} \xi \wedge \omega$,
- $(\omega \wedge \xi) \wedge \eta = \omega \wedge (\xi \wedge \eta)$.

We may assign a k-form smoothly at each point on a manifold M. We denote the space of smooth k-forms on M by $\Omega^k(M)$. Note that $\Omega^0(M)$ is $C^\infty(M)$. The exterior derivative d is defined as the map $\mathrm{d}\colon \Omega^k(M) \to \Omega^{k+1}(M)$ such that

$$\omega \mapsto \mathrm{d}\omega = \frac{1}{k!} \partial_\nu \omega_{\mu_1 \cdots \mu_k} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_k}. \tag{5}$$

c) Consider the space $\Omega^*(\mathbb{R}^3)$. Write a $\omega_k \in \Omega^k(M)$, for $1 \leq k \leq 3$, in the form (1) with the canonical coordinate system and basis $\{x, y, z; \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$ for $T^*\mathbb{R}^3$. (*2 points*)

d) Compute $\mathrm{d}\omega_k$ for each ω_k of c), i.e. for $1 \leq k \leq 3$. (*2 points*)

e) Let $\omega \in \Omega^k(M)$ and $\xi \in \Omega^l(M)$. Show that (*2 points*)

$$\mathrm{d}(\omega \wedge \xi) = \mathrm{d}\omega \wedge \xi + (-1)^k \omega \wedge \mathrm{d}\xi. \tag{6}$$

f) Let $\omega, \xi \in \Omega^k(N)$ and let $\varphi : M \to N$. Show that (*2 points*)

- $\mathrm{d}(\varphi^*\omega) = \varphi^*(\mathrm{d}\omega)$,
- $\varphi^*(\omega \wedge \xi) = (\varphi^*\xi) \wedge (\varphi^*\omega)$.

g) Prove that $\mathrm{d}^2 = 0$. (*2 points*)

h) Consider the electromagnetic potential $A = (\phi, \vec{A})$, this can be expressed as a one form $A = A_\mu \, \mathrm{d}x^\mu$. The electromagnetic tensor is defined by $F = \mathrm{d}A$. Show that $\mathrm{d}F = 0$ gives the Bianchi identity, which leads to two of the known Maxwell equations. (*1 point*)

i) Assume the manifold M is endowed with a metric g. Le us define the invariant volume element by

$$\Omega_M = \sqrt{|g|} \, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n, \tag{7}$$

where $g = \det g_{\mu\nu}$ and x^μ are the coordinates of the chart (U, ϕ). Show that (7) is, up to a sign, invariant under coordinate transformations. (*1 point*)

Moreover, it is natural to define an integration of $f \in C^\infty(M)$ over M by

$$\int_M f \Omega_M = \int_M f \sqrt{|g|} \, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n. \tag{8}$$

We note that $\dim\Omega^k(M) = \dim\Omega^{n-k}(M)$, hence $\Omega^k(M) \simeq \Omega^{n-k}(M)$. If M is endowed with a metric g, we can get a natural isomorphism between them called the Hodge operator $\star$, which defined by linear map $\star : \Omega^k(M) \to \Omega^{n-k}(M)$, with

$$\omega \mapsto \star\omega = \frac{\sqrt{|g|}}{k!(n-k)!} \omega_{\mu_1 \cdots \mu_k} \epsilon^{\mu_1 \cdots \mu_k}{}_{\nu_{k+1} \cdots \nu_n} \, \mathrm{d}x^{\nu_{k+1}} \wedge \ldots \wedge \mathrm{d}x^{\nu_n}. \tag{9}$$

We refer to $\star\omega$ as the *Hodge dual* of ω.

j) Prove that $\star\star\omega = (-1)^{k(n-k)}\omega$ if (M, g) is Riemannian and $\star\star\omega = (-1)^{1+k(n-k)}\omega$ if Lorentizian. *(3 points)*

k) What would be the Hodge dual of the electromagnetic tensor F? Prove that the rest of the Maxwell equations are given by $\mathrm{d}{\star F} = \star J$, where J is the one-form corresponding to the electromagnetic current. *(2 points)*

Let $\alpha, \beta \in \Omega^k(M)$. Since $\alpha \wedge \star\beta$ is an n-form, its integral over M is well defined. We can define the inner product of two k-forms as the bilinear map $\langle\cdot,\cdot\rangle : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}$ such that

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta. \tag{10}$$

2 Branes & M-theory (10 points)

There are a lot of motivational words attached here to what is a very simple problem; don't get too distracted. In ordinary electromagnetism with point particles, the partof the action which represents the coupling of the gauge-potential one form $A^{(1)}$ to a charged particle can be written $S = \int_\gamma A^{(1)}$, where γ is the particle worldline $\gamma : I \subset \mathbb{R} \to M$. (The superscript on $A^{(1)}$ is just to remind you that it is a one-form.) For this problem you will consider a theory related to ordinary electromagnetism, but this time in 11 spacetime dimensions, with a three-form gauge potential $A^{(3)}$ and four-form field strength $F^{(4)} = \mathrm{d}A^{(3)}$. Note that the field strength is invariant under a gauge transformation $A^{(3)} \to A^{(3)} + \mathrm{d}\lambda^{(2)}$ for any two-form $\lambda^{(2)}$.

a) What would be the number of spatial dimensions of an object to which this gauge field would naturally couple (for example, ordinary $E + M$ couples to zero-dimensional objects$\sim$point particles)? *(1 point)*

b) The electric charge of an ordinary electron is given by the integral of the dual two-form gauge field strength over a two-sphere surrounding the particle. How would you define the "charge" of the object to which $A^{(3)}$ couples? Argue that it is conserved if $\star F^{(4)} = 0$. *(3 points)*

c) Imagine there is a "dual gauge potential" $\widetilde{A}$ that satisfies $\mathrm{d}(\widetilde{A}) = \star F^{(4)}$. To what dimensionality object does it naturally couple? *(2 points)*

d) The action for the gauge field itself (as opposed to its coupling to other things) will be an integral over the entire 11-dimensional spacetime. What are the terms that would be allowed in such an action that are invariant under "local" gauge transformations, for instance, gauge transformations specified by a two-form $\lambda^{(2)}$ that vanishes at infinity? Restrict yourself to terms of first, second, or third order in $A^{(3)}$ and its first derivatives (no second derivatives, no higher-order terms). You may use the exterior derivative, wedge product, and Hodge dual, but not any explicit apperance of the metric. *(4 points)*

More background: "Supersymmetry" is a hypothetical symmetry relating bosons (particles with integral spin) and fermions (particles with spin $\frac{1}{2}, \frac{3}{2}$, etc.). An interesting feature is that supersymmetric theories are only well-defined in 11 dimensions or less–in larger number of dimensions, supersymmetry would require the existence of particles with spins greater than 2, which cannot be consistently quantized, Eleven-dimensional supersymmetry is a unique theory, which naturally includes a three-form gauge potential (not to mention gravity). "Recent" work has shown that it also includes various higher-dimensional objects alluded to in this problem (although we've cut some corners here). This theory turns out to be a well defined limit of something called M-theory, which has as other limits various 10-dimensional superstring theories.