Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 6
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SS 17

Exercises General Relativity and Cosmology

Priv.-Doz Stefan Förste, Cesar Fierro

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http://www.th.physik.uni-bonn.de/people/fierro/GRSS17/

–Homework–

1 Properties of affine Connections (16 points)

Let M be a Riemannian manifold with metric g and two charts $(U, x), (V, y)$ such that $U \cap V \neq \emptyset$. Denote the space of vector fields on M by $\mathfrak{X}(M)$. As we have seen in the lecture, an *affine connection* ∇ is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y,$$

which satisfies

- $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z,$
- $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z,$
- $\nabla_{(fX)} Y = f \nabla_X Y,$
- $\nabla_X(fY) = X[f]Y + f\nabla_X Y,$

where $X, Y, Z \in \mathfrak{X}(M)$, and $f : M \to \mathbb{R}$ is a smooth function. The connection components $\Gamma^\lambda{}_{\nu\mu}$ are given by

$$\nabla_{\partial_\nu}\partial_\mu \equiv \nabla_\nu \partial_\mu = \Gamma^\lambda{}_{\nu\mu}\partial_\lambda \tag{1}$$

Using (1) one finds that for $X^\mu\partial_\mu, Y = Y^\mu\partial_\mu$,

$$\nabla_X Y = X^\mu\Big(\frac{\partial Y^\lambda}{\partial x^\mu} + Y^\nu \Gamma^\lambda{}_{\mu\nu}\Big)\partial_\lambda \equiv X^\mu(\nabla_\mu Y)^\lambda \partial_\lambda \tag{2}$$

Now in order to define the action of the connection on general tensor fields, one first imposes the action of ∇_X on a function $f : M \to \mathbb{R}$ to be $\nabla_X f = X[f]$ and the imposes the Lebiniz rule,

$$\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2), \tag{3}$$

where $X \in \mathfrak{X}(M)$ and T_1, T_2 are tensor fields of arbitrary types.

a) Let $\omega \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$. Derive the action of an affine connection ∇ on ω,

$$(\nabla_X\omega)_\nu = X^\mu\partial_\mu\omega_\nu - X^\mu\Gamma^\lambda{}_{\mu\nu}\omega_\lambda, \tag{4}$$

by looking at $\nabla_X(\langle\omega, Y\rangle)$, where $\langle\omega, Y\rangle = \omega_\mu Y^\nu\langle \mathrm{d}x^\mu, \partial_\nu\rangle = \omega_\mu Y^\mu$. *(2 points)*

It is easy to generalize this result to tensors of arbitrary type. Let T be a (k,l) tensor, then

$$
\begin{aligned}
(\nabla_X T)^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = {} & X^\rho \partial_\rho T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + X^\rho \Gamma^{\mu_1}{}_{\rho\kappa} T^{\kappa\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + \cdots + X^\rho \Gamma^{\mu_k}{}_{\rho\kappa} T^{\mu_1\cdots\mu_{k-1}\kappa}{}_{\nu_1\cdots\nu_l} \\
& - X^\rho \Gamma^\kappa{}_{\rho\nu_1} T^{\mu_1\cdots\mu_k}{}_{\kappa\nu_2\cdots\nu_l} - \cdots - X^\rho \Gamma^\kappa{}_{\rho\nu_l} T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_{l-1}\kappa}.
\end{aligned} \tag{5}
$$

b) Consider the region $U \cap V$. Then the affine connection ∇ has components $\widetilde{\Gamma}^\gamma_{\alpha\beta}$, given by

$$\nabla_{\frac{\partial}{\partial y^\alpha}} \left(\frac{\partial}{\partial y^\beta} \right) = \widetilde{\Gamma}^\gamma_{\alpha\beta} \frac{\partial}{\partial y^\gamma}. \tag{6}$$

Show that the connection components in the coordinates y of V are related to the connection of the coordinate system x of U by the transformation

$$\widetilde{\Gamma}^\gamma{}_{\alpha\beta} = \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\gamma}{\partial y^\nu} \Gamma^\nu{}_{\lambda\mu} + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu}. \tag{7}$$

Show that this transformation rule indeed makes $\nabla_X Y$ a vector for $X, Y \in \mathfrak{X}(M)$. (*2 points*)

c) Show further, that the components for $\omega \in \Omega^1(M)$

$$(\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu} \omega_\lambda \tag{8}$$

transform as tensor components. (*1 point*)

Now we demand that the metric g be *covariantly constant*, that is, if two vectors X and Y are parallel transported, then the inner product between them remains constant under parallel transport. Let V be a tangent vector to an arbitrary curve along which the vectors are parallel transported. Then we have

$$0 = \nabla_V \Big(g(X,Y) \Big) = V^\kappa [(\nabla_\kappa g)(X,Y) + g(\nabla_\kappa X, Y) + g(X, \nabla_\kappa Y)] = V^\kappa X^\mu Y^\nu (\nabla_\kappa g)_{\mu\nu}, \tag{9}$$

where we have used that $V^\kappa \nabla_\kappa X = V^\kappa \nabla_\kappa Y = 0$. Since this is true for any curves and vectors, this means that

$$(\nabla_\kappa g)_{\mu\nu} = 0. \tag{10}$$

If the condition (10) is satisfied, the connection ∇ is said to be *metric compatible*.

d) Show that for a metric compatible connection ∇ with components $\Gamma^\lambda{}_{\mu\nu}$ the equation

$$\partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\nu} g_{\kappa\nu} - \Gamma^\kappa{}_{\lambda\nu} g_{\kappa\mu} = 0 \tag{11}$$

holds. Show that this implies

$$\Gamma^\kappa{}_{(\mu\nu)} = \widetilde{\Gamma}^\kappa{}_{\mu\nu} + \frac{1}{2}(T_\nu{}^\kappa{}_\mu + T_\mu{}^\kappa{}_\nu),$$

where $\Gamma^\kappa{}_{(\mu\nu)} = \frac{1}{2}(\Gamma^\kappa{}_{\mu\nu} + \Gamma^\kappa{}_{\nu\mu}), T^\kappa{}_{\lambda\mu} = 2\Gamma^\kappa{}_{[\lambda\mu]} = \Gamma^\kappa{}_{\lambda\mu} - \Gamma^\kappa{}_{\mu\lambda}$ and

$$\widetilde{\Gamma}^\kappa{}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{12}$$

The connection components in (12) are known as the *Christoffel symbols*. (*3 points*)

This implies, that the connection coefficients are given by

$$\Gamma^{\kappa}{}_{\mu\nu} = \widetilde{\Gamma}^{\kappa}{}_{\mu\nu} + K^{\kappa}{}_{\mu\nu}, \tag{13}$$

where $K^{\kappa}{}_{\mu\nu} \equiv \frac{1}{2}(T^{\kappa}{}_{\mu\nu} + T_{\mu}{}^{\kappa}{}_{\nu} + T_{\nu}{}^{\kappa}{}_{\mu})$ is called the *contorsion*, whereas $T^{\kappa}{}_{\mu\nu}$ is called the *torsion tensor*. This implies, that if the torsion tensor vanishes on a manifold M, the components of the metric connection ∇ are given by the Christoffel symbols. The connection is then called the *Levi-Civita connection.*

e) In the last sheet you constructed the induced metric on the two-sphere S^2 and the torus T^2 embedded in $\mathbb{R}^3$ as well as de Sitter space embedded in $\mathbb{R}^{1,4}$. They were given by

$$\begin{aligned} ds^2_{S^2} =& R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) , \\ ds^2_{T^2} =& r^2 d\theta^2 + \left(R + r\cos\theta\right)^2 d\phi^2 , \\ ds^2_{dS^4} =& -dt^2 + \alpha^2\cosh^2(t/\alpha)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] . \end{aligned} \tag{14}$$

Calculate the respective Christoffel symbols. (*8 points*)

2 Geodesic equation (5 points)

A curve is a geodesic iff there is a parametrization such that it parallel transports its own tangent vector. In the case in which the connection on the manifold is given by the Levi-Civita connection, given two points, a geodesic is also that curve $\mathcal{C}$ connecting the points, that extremizes the length functional

$$\mathrm{Length}(\mathcal{C}) = \int_{\mathcal{C}} \mathrm{d}s = \int_{\lambda_0}^{\lambda_f} \mathrm{d}\lambda \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda}}, \tag{15}$$

where λ is the parameter of the curve. For simplicity of this exercise, assume that $\mathrm{Im}\mathcal{C} \subset M$ is covered by a single chart.

a) By varying the above functional, derive the geodesic equation

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\lambda} + \Gamma^{\mu}{}_{\rho\sigma}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}\lambda} = \frac{1}{e}\frac{\mathrm{d}e}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}, \tag{16}$$

where $e = \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda}}$. (*3 points*)

b) Show that if you parametrize the curve by its proper time τ, the geodesic equation is simplified to (*2 points*)

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\lambda} + \Gamma^{\mu}{}_{\rho\sigma}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}\lambda} = 0. \tag{17}$$

Remark: Note that the variational principle, $\delta\int(g_{ij}\dot{x}^i\dot{x}^j) = 0$ gives the same geodesics as the defining property for geodesics, $\delta\int(g_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}} = 0$, where the derivative is taken with respect to any affine parameter like, for eg, the proper length. This variation gives the Christoffel connection (torsionless and metric-compatible), irrespective of any other connection that may be defined on the manifold. So, in practice, a very fast way of computing Christoffel symbols is to write down the Euler-Lagrange equations for the simplified action and then read off the Christoffel symbols from the resulting geodesic equation.

3 Proper time in general relativity and GPS (17 points)

A good approximation to the metric outside the surface of Earth is

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin\theta^2\,\mathrm{d}\phi^2)\ , \tag{18}$$

with the Newtonian gravitational potential $\Phi = -GM/r$. Here, $G = 6.67\cdot 10^{-11}\,\mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2$ is Newton's gravitational constant and $M = 5.97\cdot 10^{24}\,\mathrm{kg}$ is the mass of Earth. We will further need the radius of Earth, $R_0 = 6\,371\,\mathrm{km}$. The coordinates are chosen such that the spatial origin is located in the center of Earth and Earth rotates around the $\theta = 0$ axis.

a) For which observers is the coordinate time t the proper time? (*1 point*)

b) Consider a clock not moving relative to Earth (because it is resting on your desk). Keeping r and θ still arbitrary, calculate the time measured by this clock as a function of the elapsed coordinate time. You will see two effects, discuss these. (*4 points*)

c) Which clock runs faster: One resting on the surface of Earth or one on top of a tall building? Let us define a *second* by the requirement that a reference clock at $r = R_0$ and $\theta = \pi/2$ (equator) measures 24 h for one revolution of Earth. (*4 points*)

d) Solve for a geodesic corresponding to a circular orbit around the equator. In particular, find $\mathrm{d}\phi/\mathrm{d}t$. (*4 points*)

e) Now consider a GPS satellite orbiting at an altitude of 20 200 km above the surface of Earth around the equator. What is the time measured by a clock on this satellite needed for one complete orbit (relative to the also rotating earth)? Compare this to the time measured by the reference clock defined in item c). What are the absolute time difference and the relative deviation? (*4 points*)

For this problem you might want to consider the Christoffel symbols of (18). Up to symmetry, the non-vanishing components are given by

$$\begin{aligned}
&\Gamma^t_{tr} = -\frac{\Phi}{r}(1+2\Phi), \quad \Gamma^r_{tt} = -\frac{\Phi}{r}(1-2\Phi), \quad \Gamma^r_{rr} = \frac{\Phi}{r}(1-2\Phi), \quad \Gamma^r_{\theta\theta} = -r(1-2\Phi),\\
&\Gamma^r_{\phi\phi} = -r\sin^2\theta(1-2\Phi), \quad \Gamma^\theta_{r\theta} = r^3, \quad \Gamma^\theta_{\phi\phi} = -r^4\cos\theta\sin\theta, \quad \Gamma^\phi_{r\phi} = r^3\sin^4\theta,\\
&\Gamma^\phi_{\theta\phi} = r^4\cos\theta\sin\theta.
\end{aligned} \tag{19}$$