Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 7
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SS 17

Exercises General Relativity and Cosmology

Priv.-Doz Stefan Förste, Cesar Fierro

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Due to *Fronleichnam*, there will be NO tutorial on 15.6.2017.

http://www.th.physik.uni-bonn.de/people/fierro/GRSS17/

–HOMEWORK–

1 Locally inertial frames (8 points)

Recall from the lecture the Einstein Equivalence Principle **(EEP)**, which states:
Locally, the laws of physics reduce to special relativity.

Consider a Lorentzian manifold M with metric tensor g and a point $p \in M$. For this exercise we make first a review of the local notion of an inertial frame. This means at every $p \in M$, we should find a coordinate system such that

1. Lengths are measured locally as in Minkowksi space.
2. Locally, (infinitesimal) particles move as in Minkowski space, i.e.

$$\text{Free particles:} \quad \frac{D}{d\tau}p^\mu = \frac{\partial \varphi^\mu}{\partial \tau}(\partial_\mu p^\mu + \underbrace{\Gamma^\mu{}_{\nu\rho}p^\rho}_{=0}) = 0. \tag{1}$$

We can restate the **EEP** as follows: Given a 4d Lorentzian manifold (M, g), for any $p \in M$ there is a coordinate system (U_p, φ_p), where U_p is a neighborhood of p and $\varphi_p(p) = 0 \in \mathbb{R}^4$, such that:

- $\nabla g_p(X, Y) = g_p(\nabla X, Y) + g_p(X, \nabla Y)$: metric connection at p for any $X, Y \in T_pM$,
- $g_{\mu\nu}(p) = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$,
- $\Gamma^\nu{}_{\mu\rho}(p) = 0$.

We call such a coordinate system a *locally inertial frame*.

In the following, we want to make a construction for local inertial frames. We start with general $g_{\mu\nu}(p)$. Without loss of generality we can assume that the coordinates of p are zero.

a) Argue that there are coordinates in which $g_{\mu\nu}(p) = \eta_{\mu\nu}$. *(2 points)*

b) Change coordinates from x^μ to $x'^\mu = x^\mu + b^\mu{}_{\alpha\beta}x^\alpha x^\beta$. Show that $g'_{\mu\nu}(p) = \eta_{\mu\nu}$ still holds and find $b^\mu{}_{\alpha\beta}$ such that $\partial'_\alpha g'_{\mu\nu}(p) = 0$. This implies that all Christoffel symbols vanish and we have constructed a locally inertial frame. *(3 points)*

c) Which coordinate transformations are we now still allowed to perform such that the transformed frame is stays locally inertial? *(1 point)*

d) Can you think of another construction for a local inertial frame? Formulate your answer. *(2 points)*

2 Curvature and Riemann tensor (15 points)

As we have seen in the previous exercise sheet, the connection components $\Gamma^\mu_{\alpha\beta}$ do not transform tensorially under coordinate redefinitions. Hence once cannot expect that they have an intrinsic geometrical meaning as a measure of how much a manifold is curved. For example, on a flat space $\Gamma^\mu{}_{\alpha\beta}$ vanish for Cartesian coordinates but fail to do so in polar coordinates. Intrinsic objects that measure the curvature are the torsion tensor and the Riemann tensor. This exercise is dedicated to the Riemann tensor discussed in the lecture

$$R^\mu{}_{\alpha\beta\gamma} = \partial_\beta \Gamma^\mu{}_{\alpha\gamma} - \partial_\gamma \Gamma^\mu{}_{\alpha\beta} + \Gamma^\delta{}_{\alpha\gamma}\Gamma^\mu{}_{\delta\beta} - \Gamma^\delta{}_{\alpha\beta}\Gamma^\mu{}_{\delta\gamma}\,.$$

Note that the Riemann tensor is defined without reference to any metric and therefore the above formula holds for every connection with components $\Gamma^\mu{}_{\alpha\beta}$.

a) Consider an infinitesimal parallelogram $pqrs$ whose coordinates are x^μ, $x^\mu + \epsilon^\mu$, $x^\mu + \epsilon^\mu + \delta^\mu$ and $x^\mu + \delta^\mu$, respectively (here we assume that p,q,r,s are all covered by the same chart (U, x)). Take a vector $V_0 \in T_p(M)$, parallel transport it along the curve $C = pqr$ and call the resulting vector $V_C(r) \in T_r(M)$. Similarly, parallel transport of V_0 along $C' = psr$ yields another vector $V_{C'}(r) \in T_r(M)$. Show that the difference is given by *(4 points)*

$$V^\mu_{C'}(r) - V^\mu_C(r) = V_0^\kappa R^\mu{}_{\kappa\lambda\nu}\epsilon^\lambda\delta^\nu\,.$$

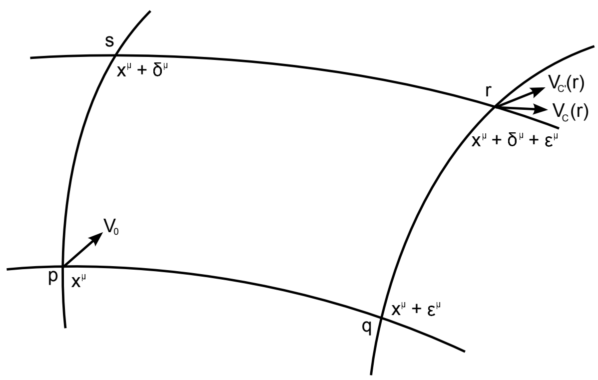

Figure 1: Infinitesimal parallelogram $pqrs$.

In the previous exercise we have seen, that it is always possible to locally find locally inertial coordinates. Note that the second derivatives of the metric do not vanish in this coordinate system![1] We will, in the following use this coordinate system to simplify some calculations. This is possible, because if one finds a purely tensorial equation, then (because of its transformation behaviour under general coordinate transformations) it is true in every coordinate system[2].

b) Consider the Riemann tensor with all indices lowered, $R_{\mu\alpha\beta\gamma} = g_{\mu\kappa} R^{\kappa}{}_{\alpha\beta\gamma}$. Use locally inertial coordinates to deduce the symmetry properties of the curvature tensor, i.e. *(2 points)*

$$\begin{aligned} R_{\kappa\lambda\mu\nu} &= -R_{\kappa\lambda\nu\mu}\,, \\ R_{\kappa\lambda\mu\nu} &= -R_{\lambda\kappa\mu\nu}\,, \\ R_{\kappa\lambda\mu\nu} &= R_{\mu\nu\kappa\lambda}\,. \end{aligned}$$

c) Show that the sum of cyclic permutations of the last three indices of the curvature tensor vanishes, i.e. *(1 point)*

$$R_{\kappa\lambda\mu\nu} + R_{\kappa\mu\nu\lambda} + R_{\kappa\nu\lambda\mu} = 0\,, \qquad 1^{\text{st}}\,\text{Bianchi identity}\,. \tag{2}$$

d) Use the results in (b)) to show that (2) is equivalent to the vanishing of the antisymmetric part of the last three indices of the Riemann tensor, *(1 point)*

$$R_{\kappa[\mu\nu\lambda]} = 0\,.$$

e) Given these relationships between the different components of the Riemann tensor, how many independent quantities remain? Deduce the number of independent components of the Riemann tensor in n dimensions. *(2 points)*

f) Make use of locally inertial coordinates once more to prove *(3 points)*

$$\nabla_{[\mu} R_{\kappa\lambda]\rho\sigma} = 0\,, \qquad 2^{\text{nd}}\,\text{Bianchi identity}\,. \tag{3}$$

g) By contracting indices of the second Bianchi identity (3) twice, show that *(2 points)*

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2}\nabla_{\nu} R\,.$$

[1] The metric at a point q near p can then be expanded as $g_{\mu\nu}(q) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\lambda\nu\rho} q^{\lambda} q^{\rho} + \ldots$. Note that in this coordinate system p has coordinates $x = (0, \ldots, 0)$.

[2] One can always construct an atlas for M with just locally inertial frames as charts. Then for two locally inertial frames $(U_p, \varphi_p), (U_q, \varphi_q)$, transform the tensorial equation on $U_p \cap U_q \neq \emptyset$.

3 Non-coordinate basis and vielbeins (5 points)

In the coordinate basis $T_p(M)$ is spanned by $\{\partial_\mu\}$ and $T_p^*(M)$ by $\{\mathrm{d}x^\mu\}$. If M is endowed with a metric g there exists an alternative choice. Consider a $\mathrm{GL}(n,\mathbb{R})$-rotation of the basis vectors ∂_μ, i.e.

$$\hat{e}_\alpha = e_\alpha{}^\mu \partial_\mu \,, \qquad (e_\alpha{}^\mu) \in \mathrm{GL}(n,\mathbb{R}) \,,$$

such that $\det(e_\alpha{}^\mu) > 0$ in order to preserve the orientation of the manifold. In addition we require $\{\hat{e}_\alpha\}$ to be orthonormal with respect to $g_{\mu\nu}$, i.e.

$$g(\hat{e}_\alpha, \hat{e}_\beta) = e_\alpha{}^\mu e_\beta{}^\nu g_{\mu\nu} = \eta_{\alpha\beta} \,.$$

If the manifold is strictly Riemannian $\eta_{\alpha\beta}$ should be replaced by $\delta_{\alpha\beta}$. Denote the inverse of $e_\alpha{}^\mu$ by $e^\alpha{}_\mu$.

a) Show that the components of a vector V in the new basis $\hat{e}_\alpha$ are related to the old components V^μ by $V^\alpha = e^\alpha{}_\mu V^\mu$. *(1 point)*

b) Introduce the dual basis $\{\hat{\theta}^\alpha\}$ to $\{\hat{e}_\alpha\}$ by $\left\langle \hat{\theta}^\alpha, \hat{e}_\beta \right\rangle = \delta^\alpha_\beta$. Conclude that $\hat{\theta}^\alpha = e^\alpha{}_\mu \, \mathrm{d}x^\mu$. *(2 points)*

$\{\hat{e}_\alpha\}$ and $\{\hat{\theta}^\alpha\}$ are called the *non-coordiante basis* and $e^\alpha{}_\mu$ are called the *vielbeins*.

c) Show that the metric is given by $\mathrm{d}s^2 = \eta_{\alpha\beta}\, \hat{\theta}^\alpha \otimes \hat{\theta}^\beta$. *(1 point)*

d) Consider the standard induced metric on S^2 as in H 9.1. Calculate the non-coordinate basis $\hat{\theta}^\alpha$ as well as the *zweibeins* $e^\alpha{}_\mu$. *(1 point)*

The non-coordinate basis is of great interest in general relativity, because it allows for the definition of spinors on curved spacetimes[3].

[3] The curved spacetime counterparts to the γ-matrices in flat spacetime are defined as $\gamma^\mu = e_\alpha{}^\mu \gamma^\alpha$ and fulfill $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.